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A

TREATISE
OF
ALGEBRA,
IN
THREE PARTS.

CONTAINING

I. The fundamental Rules and Operations;

II. The Composition and Resolution of Equations of all Degrees; and the different Affections of their Roots.

III. The Application of Algebra and Geometry to each other.

To which is added an

APPENDIX,

Concerning the general Properties
of Geometrical Lines.

By COLIN MACLAURIN, M. A.

K

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Edinburgh, and Fellow of the Royal Society.

The THIRD EDITION.

LONDON,
Printed for J. Nourse, W. Strahan, J. and F. Rivington,
W. Johnston, T. Longman, Robinson and Roberts,
S. Bladon, and T. Cadell. 1771.

BRITISH MUSEUM

TO
HIS GRACE
THOMAS,
LORD ARCHBISHOP
OF
CANTERBURY,
Primate of all ENGLAND:

As a teſtimony of GRATITUDE for the friendſhip and generous protection, with which HIS GRACE was pleaſed to honour my deceaſed huſband,

THIS
TREATISE
IS INSCRIBED
BY HIS GRACE'S
MOST OBLIGED
AND
OBEDIENT HUMBLE SERVANT,

ANNE MACLAURIN.

BRITISH MUSEUM

TO THE

READER.

THAT Mr. MACLAURIN *had, many years ago, intended to publiſh a Treatiſe on this ſubject, appears from a letter, dated April* 19, 1729, *to his honoured friend* MARTIN FOLKES, *Eſq; now Preſident of the Royal Society**. *And we find, in one of his manuſcripts, the Plan of ſuch a Work, agreeing, almoſt in every article, with the contents of this Volume.*

Had the celebrated Author lived to publiſh his own Work, his name would, alone, have been ſufficient to recommend it to the notice of the Publick: but that Taſk having, by his lamented premature death, devolved to the gentlemen whom he left entruſted with his Papers, the Reader may reaſonably expect ſome account of the materials of which it conſiſts, and of the care that has been taken in collecting and diſpoſing them, ſo as beſt to anſwer the Author's intention, and fill up the Plan he had deſigned.

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* Phil. Tranſ. Nº 408.

He ſeems, in compoſing this Treatiſe, to have had theſe three Objects in view.

1. *To give the general* Principles *and* Rules *of the Science, in the ſhorteſt, and, at the ſame time, the moſt clear and comprehenſive manner that was poſſible. Agreeable to this, though every Rule is properly exemplified, yet he does not launch out into what we may call, a* Tautology *of examples. He rejects ſome applications of* Algebra, *that are commonly to be met with in other writers; becauſe the number of ſuch applications is endleſs: and, however uſeful they may be in Practice, they cannot, by the rules of good* method, *have place in an* elementary *Treatiſe. He has likewiſe omitted the Algebraical ſolution of* particular *Geometrical problems, as requiring the knowledge of the Elements of* Geometry; *from which thoſe of* Algebra *ought to be kept, as they really are, entirely diſtinct; reſerving to himſelf to treat of the mutual relation of the two Sciences in his* Third Part, *and, more generally ſtill, in the* Appendix. *He might think too, that ſuch an application was the leſs neceſſary, that* Sir ISAAC NEWTON'*s excellent Collection of Examples is in every body's hands, and that there are few Mathematical writers,*

who do not furniſh numbers of the ſame kind.

2. *Sir* ISAAC NEWTON'*s Rules, in his* Arithmetica Univerſalis, *concerning the Reſolution of the higher equations, and the Affections of their roots, being, for the moſt part, delivered without any demonſtration, Mr.* MACLAURIN *had deſigned, that his Treatiſe ſhould ſerve as a Commentary on that Work. For we here find all thoſe difficult paſſages in Sir* ISAAC'*s Book, which have ſo long perplexed the Students of Algebra, clearly explained and demonſtrated. How much ſuch a Commentary was wanted, we may learn from the words of a late eminent Author* *. "The ableſt Mathematicians of the laſt age "*(ſays he)* did not diſdain to write Notes on "the Geometry of DES CARTES; and ſurely Sir ISAAC NEWTON's Arithmetick no "leſs deſerves that honour. To excite ſome "one of the many ſkilful Hands that our "times afford to undertake this Work, and "to ſhew the Neceſſity of it, I give this "*Specimen*, in an explication of two paſ-

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* *s'Graveſande* in Præfat. ad Specimen Comment. in Arith. Univerſ.

"ſages * of the *Arithmetica Univerſalis*; "which, however, are not the moſt difficult in that Book."

What this learned Profeſſor ſo earneſtly wiſhed for, we at laſt ſee executed; not ſeparately, nor in the looſe diſagreeable form which ſuch Commentaries generally take, but in a manner equally natural and convenient; every Demonſtration being aptly inſerted into the Body of the Work, as a neceſſary and inſeparable Member; an Advantage which, with ſome others, obvious enough to an attentive Reader, will, it is hoped, diſtinguiſh this Performance from every other, of the kind, that has hitherto appeared.

3. *After having fully explained the Nature of Equations, and the Methods of finding their Roots, either infinite expreſſions, when it can be done, or in infinite* converging ſeries; *it remained only to conſider the Relation of* Equations *involving two variable quantities, and of Geometrical* Lines *to each other; the Doctrine of the* Loci; *and the* Conſtruction *of Equations. Theſe make the Subject of the* Third Part.

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* *Viz.* The finding of *Diviſors*, and the evolution of *Binomial Surds*. See § 59—72. Part II. § 127. Part I.

Upon this Plan Mr. MACLAURIN *compoſed a ſyſtem of* Algebra, *ſoon after his being choſen Profeſſor of Mathematicks in the Univerſity of* Edinburgh; *which he, thenceforth, made uſe of in his ordinary Courſe of Lectures, and was occaſionally improving to the Perfection he intended it ſhould have, before he committed it to the Preſs. And the beſt Copies of his Manuſcript having been tranſmitted to the Publiſher, it was eaſy, by comparing them, to eſtabliſh a correct and genuine Text. There were, beſides, ſeveral detached Papers, ſome of which were quite finiſhed, and wanted only to be inſerted in their proper places. In a few others, the Demonſtrations were ſo conciſely expreſſed, and couched in Algebraical characters, that it was neceſſary to write them out at more length, to make them of a piece with the reſt. And this is the only liberty the Publiſher has allowed himſelf to take; excepting a few inconſiderable additions, that ſeemed neceſſary to render the Book more compleat within itſelf, and to ſave the trouble of conſulting others who have written on the ſame Subject.*

The Rules concerning the Impoſſible roots *of Equations, our Author had very fully conſidered, as appears from his Manuſcript papers: but as he had no where reduced any thing on* *that Subject to a better form, than what was long ago publiſhed in the* Philoſophical Tranſactions, N° 394, *and* 408. *we thought it beſt to take the ſubſtance of* Chap. 11. Part II. *from thence; eſpecially as the latter of theſe Papers furniſhes a demonſtration of the original Rule, which pre-ſuppoſes only what the Reader has been taught in a preceding Chapter.*

The Paper that is ſubjoined, on the Sums of the Powers of the Roots of an Equation, *is taken from a Letter of the Author* (8 Jul. 1743) *to the Right Honourable the Earl* STANHOPE; *communicated to the Publiſher, with ſome things added by his Lordſhip, which were wanting to finiſh the Demonſtration.*

Of theſe Materials, carefully collected and put in order, the following Elementary Treatiſe *is compoſed; which we have choſen rather to give in a Volume that is within the reach of every Student, than in one more pompous, which might be leſs generally uſeful. And we hope, from the pains it has coſt us, its blemiſhes are not many, nor ſuch as a candid intelligent Reader may not forgive.*

The Latin Appendix *is a proper Sequel, and a high Improvement, of what had been demonſtrated in* Part. III. *concerning the Relation of* Curve lines *and* Equations; *a Subject which* *our Author had been early and intimately acquainted with; witneſs his* Geometria Organica, *printed in* 1719, *when he was not full twenty-one years of age, and which, though ſo juvenile a work, gained him, at once, that diſtinguiſhed Rank among Mathematicians, which he thenceforth held with great luſtre. Yet he frankly owns, he was led to many of the Propoſitions in this* Appendix, *from a Theorem of Mr.* COTES, *communicated to him, without any demonſtration, by the Reverend and Learned Dr.* SMITH, *Maſter of* Trinity-College, Cambridge. *How he has profited of that Hint, the Learned will judge: Thus much we can venture to ſay, that he himſelf ſet ſome value upon this Performance; having, we are told, employed ſome of the lateſt hours he could give to ſuch Studies, in reviſing it for the Preſs; to bequeath it as his laſt Legacy to the Sciences and to the Publick.*

The gentlemen to whom Mr. MACLAURIN *left the care of his Papers, are* MARTIN FOLKES, *Eſq; Preſident of the Royal Society;* ANDREW MITCHEL, *Eſq; and the Reverend Mr.* HILL, *Chaplain to his Grace the Archbiſhop of* Canterbury; *with whom he had lived in a moſt intimate friendſhip. And by their direction this Treatiſe is publiſhed.*

17 AP 66

CONTENTS.

PART I.

Chap. I. *DEFINITIONS and Illuſtrations* Pag. 1
II. *Of Addition* 8
III. *Of Subtraction* 11
IV. *Of Multiplication* 12
V. *Of Diviſion* 17
VI. *Of Fractions* 24
VII. *Of the Involution of quantities* 34
VIII. *Of Evolution* 42
IX. *Of Proportion* 54
X. *Of Equations that involve only one unknown quantity* 61
XI. *Of the Solution of queſtions that produce ſimple equations* 68
XII. *Containing ſome general Theorems for the exterminating unknown quantities in given equations* 81
XIII. *Of Quadratic equations* 85
XIV. *Of Surds* 94
Supplement to Chap. XIV. 127

CONTENTS.

PART II.

Of the Geneſis and Reſolution of equations of all degrees; and of the different Affections of the roots.

Chap. I. *Of the Geneſis and Reſolution of equations in general; and the number of roots an equation of any degree may have* Pag. 131

II. *Of the Signs and Coefficients of equations* 139

III. *Of the Transformation of equations, and exterminating their intermediate terms* 148

IV. *Of finding the Roots of equations when two or more of the roots are equal to each other* 162

V. *Of the Limits of equations* 170

VI. *Of the Reſolution of equations, all whoſe roots are commenſurate* 186

VII. *Of the Reſolution of equations by finding the equations of a lower degree that are their diviſors* 197

Supplement to Chap. VII. *Of the Reduction of equations by ſurd diviſors* 213

VIII. *Of the Reſolution of equations by* Cardan's *rule, and others of that kind* 222

IX. *Of the Methods by which you may approximate to the roots of* nu-

meral

meral *equations by their limits* Pag. 230

Chap. X. *Of the Method of* Series, *by which you may approximate to the roots of* literal *equations* 243

XI. *Of the Rules for finding the number of impoſſible roots in an equation.* 274

XII. *Containing a general Demonſtration of Sir* Iſaac Newton'*s rule for finding the ſums of the powers of the roots of an equation* 285

PART III.

Of the Application of *Algebra* aud *Geometry* to each other.

Chap. I. *Of the Relation between the equations of curve lines and the figure of thoſe curves, in general* 296

II. *Of the* Conſtruction *of* quadratic equations, *and the Properties of the lines of the ſecond order* 324

III. *Of the Conſtruction of* cubic *and* biquadratic *equations* 351

APPENDIX.

De Linearum Geometricarum Proprietatibus generalibus.

17 AP 66

A TREATISE OF ALGEBRA.

PART I.

CHAP. I.

Deſinitions and Illuſtrations.

§ 1. ALGEBRA is a general method of computation by certain ſigns and ſymbols which have been contrived for this purpoſe, and found convenient. It is called an UNIVERSAL ARITHMETICK, and proceeds by operations and rules ſimilar to thoſe in common arithmetick, founded upon the ſame principles. This, however, is no argument againſt its uſefulneſs or evidence; ſince arithmetick is not to be the leſs valued that it is common, and is allowed to be one of the moſt clear and evident of the ſciences. But as a number of ſymbols are admited into this ſcience, being neceſſary for giving it that extent and generality which is its greateſt excellence; the import of thoſe ſymbols is to be clearly ſtated, that no obſcurity or error may ariſe from the frequent uſe and complication of them.

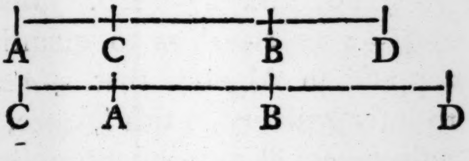
§ 2. In GEOMETRY, lines are repreſented by a line, triangles by a triangle, and other figures by a figure of the ſame kind; but, in ALGEBRA, quantities are repreſented by the ſame letters of the alphabet; aud various ſigns have been imagined for repreſenting their affections, relations, and dependencies. In Geometry the repreſentations are more natural, in Algebra more arbitrary: the former are like the firſt attempts towards the expreſſion of objects, which was by drawing their reſemblances; the latter correſpond more to the preſent uſe of languages and writing. Thus the evidence of Geometry is ſometimes more ſimple and obvious; but the uſe of Algebra more extenſive, and often more ready: eſpecially ſince the mathematical ſciences have acquired ſo vaſt an extent, and have been applied to ſo many enquiries.

§ 3. In thoſe ſciences, it is not barely magnitude that is the object of contemplation: but there are many affections and properties of quantities, and operations to be performed upon them, that are neceſſarily to be conſidered. In eſtimating the ratio or proportion of quantities, magnitude only is conſidered (*Elem.* 5. *Def.* 3.) But the nature and properties of figures depend on the poſition of the lines that bound them, as well as on their magnitude. In treating of motion, the direction of motion as well as its velocity; and the direction of powers that generate or deſtroy motion, as well as their forces, muſt be regarded. In OPTICS, the poſition, brightneſs, and diſtinctneſs of images, are of no leſs importance than their bigneſs; and the like is to be ſaid of other ſciences. It is neceſſary therefore that other ſymbols be admitted into Algebra beſide the letters and numbers which repreſent the magnitude of quantities.

§ 4. The relation of equality is expreſſed by the ſign $=$; thus to expreſs that the quantity repreſented by a is equal to that which is repreſented by b, we write $a=b$. But if we would expreſs that a is greater than b, we write $a \sqsubset b$; and if we would expreſs algebraically that a is leſs than b, we write $a \sqsupset b$.

§ 5. QUANTITY is what is made up of parts, or is capable of being greater or leſs. It is increaſed by *Addition*, and diminiſhed by *Subtraction*; which are therefore the two primary operations that relate to quantity. Hence it is, that any quantity may be ſuppoſed to enter into algebraic computations two different ways which have contrary effects; either as an *increment* or as a *decrement*; that is, as a quantity to be added or as a quantity to be ſubtracted. The ſign + (*plus*) is the mark of *Addition*, and the ſign − (*minus*) of *Subtraction*. Thus the quantity being repreſented by a, $+a$ imports that a is to be added, or repreſents an increment; but $-a$ imports that a is to be ſubtracted, and repreſents a decrement. When ſeveral ſuch quantities are joined, the ſigns ſerve to ſhew which are to be added, and which are to be ſubtracted. Thus $+a+b$ denotes the quantity that ariſes when a and b are both conſidered as increments, and therefore expreſſes the ſum of a and b. But $+a-b$ denotes the quantity that ariſes when from the quantity a the quantity b is ſubtracted; and expreſſes the exceſs of a above b. When a is greater than b, then $a-b$ is itſelf an increment; when $a=b$, then $a-b=0$; and when a is leſs than b, then $a-b$ is itſelf a decrement.

§ 6. As addition and ſubtraction are oppoſite, or an increment is oppoſite to a decrement, there is an analogous oppoſition between the affections of quantities that are conſidered in the mathematical ſciences. As between exceſs and defect; between the value of effects or money due to a man, and money due by him; a line drawn towards the right, and a line drawn to the left; gravity and levity; elevation above the horizon, and depreſſion below it. When two quantities equal in reſpect of magnitude, but of thoſe oppoſite kinds, are joined together, and conceived to take place in the ſame ſubject, they deſtroy each other's effect, and their amount is *nothing*. Thus 100 *l.* due to a man and 100 *l.* due by him balance each other, and in eſtimating his ſtock may be both neglected. Power is ſuſtained by an equal power acting on the ſame body with a contrary direction, and neither have effect. When two unequal quantities of thoſe oppoſite qualities are joined in the ſame ſubject, the greater prevails by their difference. And when a greater quantity is taken from a leſſer of the ſame kind, the remainder becomes of the oppoſite kind. Thus if we add the lines AB and BD together, their ſum is AD; but if we are to ſubtract B D from AB, then BC=BD is to be taken the contrary way towards A, and the remainder is AC; which, when BD, or BC exceeds AB, becomes a line on the other ſide of A. When two powers or forces are to be added together, their ſum acts



upon the body: but when we are to ſubtract one of them from the other, we conceive that which is to be ſubtracted to be a power with an oppoſite direction; and if it be greater than the other, it will prevail by the difference. This change of quality however only takes place where the quantity is of ſuch a nature as to admit of ſuch a contrariety or oppoſition. We know nothing analogous to it in quantity abſtractly conſidered; and cannot ſubtract a greater quantity of matter from a leſſer, or a greater quantity of light from a leſſer. And the application of this doctrine to any art or ſcience is to be derived from the known principles of the ſcience.

§ 7. A quantity that is to be added is likewiſe called a *poſitive* quantity; and a quantity to be ſubtracted is ſaid to be *negative:* they are equally real, but oppoſite to each other, ſo as to take away each other's effect, in any operation, when they are equal as to quantity. Thus $3-3=0$, and $a-a=0$. But though $+a$ and $-a$ are equal as to quantity, we do not ſuppoſe in Algebra that $+a=-a$; becauſe to infer equality in this ſcience, they muſt not only be equal as to quantity, but of the ſame quality, that in every operation the one may have the ſame effect as the other. A decrement may be equal to an increment, but it has in all operations a contrary effect; a motion downwards may be equal to a motion upwards, and the depreſſion of a ſtar below the horizon may be equal to the elevation of a ſtar above it: but thoſe poſitions are oppoſite, and the diſtance of the ſtars is greater than if one of them was at the horizon ſo as to have no elevation above it, or depreſſion below it. It is on account of this contrariety that a negative quantity is ſaid to be leſs than nothing, becauſe it is oppoſite to the poſitive, and diminiſhes it when joined to it, whereas the addition of 0 has no effect. But a negative is to be conſidered no leſs as a real quantity than the poſitive. Quantities that have no ſign prefixed to them are underſtood to be poſitive.

§ 8. The number prefixed to a letter is called the numeral *coefficient*, and ſhews how often the quantity repreſented by the letter is to be taken. Thus $2a$ imports that the quantity repreſented by a is to be taken twice; $3a$ that it is to be taken thrice; and ſo on. When no number is prefixed, *unit* is underſtood to be the coefficient. Thus 1 is the coefficient of a or of b.

Quantities are ſaid to be *like* or *ſimilar*, that are repreſented by the ſame letter or letters equally repeated. Thus $+3a$ and $-5a$ are like; but a and b, or a and aa are unlike.

A quantity is ſaid to conſiſt of as many *terms* as there are parts joined by the ſigns +

or $-$; thus $a+b$ conſiſts of two terms, and is called a *binomial*; $a+b+c$ conſiſts of three terms, and is called a *trinomial.* Theſe are called *compound* quantities: a *ſimple* quantity conſiſts of one term only, as $+a$, or $+ab$, or $+abc$.

The other ſymbols and definitions neceſſary in *Algebra* ſhall be explained in their proper places.

CHAP. II.

Of ADDITION.

§ 9. CASE I. To add quantities that are like and have like ſigns.

Rule. *Add together the coefficients, to their ſum prefix the common ſign, and ſubjoin the common letter or letters.*

EXAMPLES.

To	$+5a$	to	$-6b$	to	$a+b$
Add	$+4a$	add	$-2b$	add	$3a+5b$
Sum	$+9a$	Sum	$-8b$	Sum	$4a+6b$

$$\begin{array}{ll} \text{To} & 3a-4x \\ \text{Add} & 5a-8x \\ \hline \text{Sum} & 8a-12x \end{array}$$

Caſe II. To add quantities that are like but have unlike ſigns.

Rule. *Subtract the leſſer coefficient from the greater, prefix the ſign of the greater to the remainder, and ſubjoin the common letter or letters.*

EXAMPLES.

$$\begin{array}{ll|r} \text{To} & -4a & +5b-6c \\ \text{Add} & +7a & -3b+8c \\ \hline \text{Sum} & +3a & 2b+2c \end{array}$$

$$\begin{array}{ll|r} \text{To} & a+6x-5y+8 & 2a-2b \\ \text{Add} & -5a-4x+4y-3 & -2a+2b \\ \hline \text{Sum} & -4a+2x-y+5 & 0\,..\,0 \end{array}$$

This rule is eaſily deduced from the nature of poſitive and negative quantities.

If there are more than two quantities to be added together, firſt add the poſitive together into one ſum, and then the negative (by *Caſe* I.) Then add theſe two ſums together (by *Caſe* II.)

EXAMPLE.

$$\begin{array}{lr}
 & +\ 8a \\
 & -\ 7a \\
 & +10a \\
 & -12a \\
\hline
\text{Sum of the poſitive} \ldots & +18a \\
\text{Sum of the negative} \ldots & -19a \\
\hline
\text{Sum of all} \ldots\ldots & -\ a
\end{array}$$

Caſe III. To add quantities that are unlike.

Rule. *Set them all down one after another, with their ſigns and coefficients prefixed.*

EXAMPLES.

$$\begin{array}{ll|r}
\text{To} & +2a & +3a \\
\text{Add} & +3b & -4x \\
\hline
\text{Sum} & 2a+3b & 3a-4x
\end{array}$$

$$\begin{array}{ll}
\text{To} & 4a+4b+3c \\
\text{Add} & -4x-4y+3z \\
\hline
\text{Sum} & 4a+4b+3c-4x-4y+3z
\end{array}$$

CHAP. III.

Of SUBTRACTION.

§ 10. GEnerale rule. *Change the ſigns of the quantity to be ſubtracted into their contrary ſigns, and then add it ſo changed to the quantity from which it was to be ſubtracted (by the rules of the laſt chapter:) the ſum ariſing by this addition is the remainder.* For, to ſubtract any quantity, either poſitive or negative, is the ſame as to add the oppoſite kind.

EXAMPLES.

$$\begin{array}{lr|r} \text{From} & +5a & 8a-7b \\ \text{Subtract} & +3a & 3a+4b \\ \hline \text{Remaind.} & 5a-3a, \text{ or } 2a & 5a-11b \end{array}$$

$$\begin{array}{lr} \text{From} & 2a-3x+5y-6 \\ \text{Subtract} & 6a+4x+5y+4 \\ \hline \text{Remaind.} & -4a-7x\quad 0\ -10 \end{array}$$

It is evident, that to ſubtract or take away a decrement is the ſame as adding an equal increment. If we take away $-b$ from $a-b$, there remains a; and if we add $+b$ to $a-b$, the ſum is likewiſe a. In general, the ſubtraction of a negative quantity is equivalent to adding its poſitive value.

CHAP.

CHAP. IV.

Of MULTIPLICATION.

§ 11. IN Multiplication the General rule for the ſigns is, That *when the ſigns of the factors are like* (i. e. both +, or both −,) *the ſign of the product is* + ; but *when the ſigns of the factors are unlike, the ſign of the product is* —.

Caſe I. When any poſitive quantity, $+a$, is multiplied by any poſitive number, $+n$, the meaning is, That $+a$ is to be taken as many times as there are units in n; and the product is evidently na

Caſe II. When $-a$ is multiplied by n, then $-a$ is to be taken as often as there are units in n, and the product muſt be $-na$.

Caſe III. Multiplication by a poſitive number implies a repeated addition: but multiplication by a negative implies a repeated ſubtraction. And when $+a$ is to be multiplied by $-n$, the meaning is, That $+a$ is to be ſubtracted as often as there are units in n: therefore

fore the product is negative, being $-na$.

Caſe IV. When $-a$ is to be multiplied by $-n$, then $-a$ is to be ſubtracted as often as there are unites in n; but (by § 10.) to ſubtract $-a$ is equivalent to adding $+a$, conſequently the product is $+na$.

The IId and IVth caſes may be illuſtrated in the following manner.

By the definitions, $+a-a=0$; therefore, if we multitply $+a-a$ by n, the product muſt vaniſh or be 0, becauſe the factor $a-a$ is 0. The firſt term of the product is $+na$ (by Caſe I.) Therefore the ſecond term of the product muſt be $-na$, which deſtroys $+na$; ſo that the whole product muſt be $+na-na=0$. Therefore $-a$ multiplied by $+n$ gives $-na$.

In like manner, if we multiply $+a-a$ by $-n$, the firſt term of the product being $-na$, the latter term of the product muſt be $+na$, becauſe the two together muſt deſtroy each other, or their amount be 0, ſince one of the factors (*viz.* $a-a$) is 0. Therefore $-a$ multiplied by $-n$ muſt give $+na$.

In this general doctrine the multiplicator is always conſidered as a number. A quantity of any kind may be multiplied by a number: but a pound is not to be multiplied by a pound, or a debt by a debt, or a line by a line. We ſhall afterwards conſider the analogy there is betwixt rectangles in Geometry and a product of two factors.

§ 12. If the quantities to be multiplied are *ſimple* quantities, *find the ſign of the product by the laſt rule; after it place the product of the coefficients, and then ſet down all the letters after one another as in one word.*

EXAMPLES.

Mult.	$+a$	$-2a$	$6x$
By	$+b$	$+4b$	$-5a$
Prod.	$+ab$	$-8ab$	$-30ax$

Mult.	$-8x$	$+3ab$
By	$-4a$	$-5ac$
Prod.	$+32ax$	$-15aabc$

§ 13. To multiply *compound* quantities, you muſt *multiply every part of the multiplicand by all the parts of the multiplier taken one after another, and then collect all the products into one ſum: that ſum ſhall be the product required.*

EXAMPLES.

$$\begin{array}{ll|l}
\text{Mult.} & a+b & 2a-3b \\
\text{By} & a+b & 4a+5b \\
\cline{2-3}
\text{Prod.} & \left\{\begin{array}{l} aa+ab \\ \quad +ab+bb \end{array}\right. & \left\{\begin{array}{l} 8aa-12ab \\ \quad +10ab-15bb \end{array}\right. \\
\cline{2-3}
\text{Sum} & aa+2ab+bb & 8aa-2ab-15bb
\end{array}$$

$$\begin{array}{ll|l}
\text{Mult.} & 2a-4b & xx-ax \\
\text{By} & 2a+4b & x+a \\
\cline{2-3}
\text{Prod.} & \left\{\begin{array}{l} 4aa-8ab \\ \quad +8ab-16bb \end{array}\right. & \left\{\begin{array}{l} xxx-axx \\ \quad +axx-aax \end{array}\right. \\
\cline{2-3}
\text{Sum} & 4aa\,..\,0-16bb & xxx\,..\,0-aax
\end{array}$$

$$\begin{array}{ll}
\text{Mult.} & aa+ab+bb \\
\text{By} & a-b \\
\cline{2-2}
\text{Prod.} & \left\{\begin{array}{l} aaa+aab+abb \\ \quad -aab-abb-bbb \end{array}\right. \\
\cline{2-2}
\text{Sum} & aaa\,..\,0\,...\,0-bbb
\end{array}$$

§ 14. Products that ariſe from the multiplication of two, three, or more quantities, as *abc*, are ſaid to be of two, three, or more *dimenſions*; and thoſe quantities are called *factors* or *roots*.

If all the factors are equal, then theſe products are called *powers*; as aa or aaa are powers of a. Powers are expreſſed ſometimes by placing above the root to the right-hand a figure expreſſing the number of factors that produce them. Thus,

a	is called the	1ſt	Power of the root a, and is ſhortly expreſſed thus,	a
aa		2d		a^2
aaa		3d		a^3
$aaaa$		4th		a^4
$aaaaa$		5th		a^5

§ 15. Theſe figures which expreſs the number of factors that produce powers are called their *indices* or *exponents*; thus 2 is the index of a^2. And *powers of the ſame root are multiplied by adding their exponents.* Thus $a^2 \times a^3 = a^5$, $a^4 \times a^3 = a^7$, $a^3 \times a = a^4$.

§ 16. Sometimes it is uſeful not actually to multiply compound quantities, but to ſet them down with the ſign of multiplication (×) between them, drawing a line over each of the compound factors. Thus $\overline{a+b} \times \overline{a-b}$ expreſſes the product of $a+b$ multiplied by $a-b$.

CHAP.

CHAP. V.

Of DIVISION.

§ 17. THE ſame rule for the ſigns is to be obſerved in Diviſion as in Multiplication; that is, *If the ſigns of the dividend and diviſor are like, the ſign of the quotient muſt be +; if they are unlike, the ſign of the quotient muſt be —.* This will be eaſily deduced from the rule in Multiplication, if you conſider that the quotient muſt be ſuch a quantity as multiplied by the diviſor ſhall give the dividend.

§ 18. The General rule in Diviſion is, *to place the dividend above a ſmall line, and the diviſor under it, expunging any letters that may be found in all the quantities of the dividend and diviſor, and dividing the coefficients of all the terms by any common meaſure.* Thus when you divide $10ab+15ac$ by $20ad$, expunging a out of all the terms, and dividing all the coefficients by 5, the quotient is $\frac{2b+3c}{4d}$; and

$$2b)\,ab+bb\,\Big(\frac{a+b}{2}.$$

$$12ab)\,30ax-54ay\,\Big(\frac{5x-9y}{2b}.$$

$$4aa)\;8ab+6ac\;\Big(\frac{4b+3c}{2a}.$$

$$\text{And }2bc)\;5abc\;\Big(\frac{5a}{2}.$$

§ 19. *Powers of the ſame root are divided by ſubtracting their exponents as they are multiplied by adding them.* Thus if you divide a^5 by a^2, the quotient is a^{5-2} or a^3. And b^6 divided by b^4 gives b^{6-4} or b^2; and a^7b^5 divided by a^2b^3 gives a^5b^2 for the quotient.

§ 20. *If the quantity to be divided is compound, then you muſt range its parts according to the dimenſions of ſome one of its letters, as in the following example.* In the dividend $a^2+2ab+b^2$, they are ranged according to the dimenſions of a, the quantity a^2 where a is of two dimenſions being placed firſt, $2ab$ where it is of one dimenſion next, and b^2, where it is not at all, being placed laſt. *The diviſor $a+b$, muſt be ranged according to the dimenſions of the ſame letters; then you are to divide the firſt term of the dividend by the firſt term of the diviſor, and to ſet down the quotient, which, in this example, is a; then multiply this quotient by the whole diviſor, and ſubtract the product from the dividend, and the remainder ſhall give a new dividend, which in this example is $ab+b^2$.*

$a+b)$

$$\begin{array}{l}
a+b)\,a^2+2ab+b^2\,(a+b \\
\underline{a^2+\;ab} \\
\,ab+b^2 \\
\,\underline{ab+b^2} \\
\,\;0\quad 0
\end{array}$$

Divide the firſt term of this new dividend by the firſt term of the diviſor and ſet down the quotient (which in this example is b) with its proper ſign. Then multiply the whole diviſor by this part of the quotient, and ſubtract the product from the new dividend; and if there is no remainder, the diviſion is finiſhed: If there is a remainder, you are to proceed after the ſame manner till no remainder is left; or till it appear that there will be always ſome remainder.

Some Examples will illuſtrate this operation.

EXAMPLE I.

$$\begin{array}{l}
a+b)\,a^2-b^2\,(a-b \\
\underline{a^2+ab} \\
-ab-b^2 \\
\underline{-ab-b^2} \\
0\quad 0
\end{array}$$

EXAMPLE II.

$$\begin{array}{l} a-b)\,aaa-3aab+3abb-bbb\,(aa-2ab+bb \\ aaa-\;aab \\ \hline -2aab+3abb-bbb \\ -2aab+2abb \\ \hline abb-bbb \\ abb-bbb \\ \hline 0\quad\;\;0 \end{array}$$

EXAMPLE III.

$$\begin{array}{l} a-b)\,aaa-bbb\,(aa+ab+bb \\ aaa-aab \\ \hline aab-bbb \\ aab-abb \\ \hline abb-bbb \\ abb-bbb \\ \hline 0\quad\;\;0 \end{array}$$

EXAMPLE IV.

$$\begin{array}{rlllll}
3a-6) & 6aaaa-96\ (2aaa+4aa+8a+16 \\
 & \underline{6aaaa-12aaa} \\
 & \quad\quad 12aaa-96 \\
 & \quad\quad \underline{12aaa-24aa} \\
 & \quad\quad\quad\quad 24aa-96 \\
 & \quad\quad\quad\quad \underline{24aa-48a} \\
 & \quad\quad\quad\quad\quad\quad 48a-96 \\
 & \quad\quad\quad\quad\quad\quad \underline{48a-96} \\
 & \quad\quad\quad\quad\quad\quad\ \ 0 \quad\ \ 0
\end{array}$$

§ 21. It often happens that the operation may be continued without end, and then you have an *infinite Series* for the quotient; and *by comparing the firſt three or four terms you may find what law the terms obſerve: by which means, without any more diviſion, you may continue the quotient as far as you pleaſe.* Thus, in dividing 1 by $1-a$, you find the quotient to be $1+a+aa+aaa+aaaa+$ &c. which Series can be continued as far as you pleaſe by adding the powers of a.

The operation is thus:

$$\begin{array}{l}
1-a)\ 1\ (1+a+aa+aaa,\ \&c.\\
\qquad\quad \underline{1-a}\\
\qquad\qquad +a\\
\qquad\qquad \underline{+a-aa}\\
\qquad\qquad\qquad +aa\\
\qquad\qquad\qquad \underline{+aa-aaa}\\
\qquad\qquad\qquad\qquad +aaa\\
\qquad\qquad\qquad\qquad \underline{+aaa-aaaa}\\
\qquad\qquad\qquad\qquad\qquad +aaaa,\ \&c.
\end{array}$$

Another Example.

$$\begin{array}{l}
a+x)\ aa+xx\ (a-x+\frac{2xx}{a}-\frac{2x^3}{a^2}+\frac{2x^4}{a^3}-\&c.\\
\qquad\quad \underline{aa+ax}\\
\qquad\qquad -ax+xx\\
\qquad\qquad \underline{-ax-xx}\\
\qquad\qquad\qquad +2xx\\
\qquad\qquad\qquad \underline{+2xx-\frac{2x^3}{a}}\\
\qquad\qquad\qquad\qquad -\frac{2x^3}{a}\\
\qquad\qquad\qquad\qquad \underline{-\frac{2x^3}{a}-\frac{2x^4}{a^2}}\\
\qquad\qquad\qquad\qquad\qquad +\frac{2x^4}{a^2}\\
\qquad\qquad\qquad\qquad\qquad \&c.
\end{array}$$

In this laſt example the ſigns are alternately + and —, the coefficient is conſtantly 2, after the firſt two terms, and the letters are the powers of x and a; ſo that the quotient may be continued as far as you pleaſe without any more diviſion.

But in Diviſion, after you come to a remainder of one term, as $2xx$ in the laſt example, it is commonly ſet down with the diviſor under it, after the other terms, and theſe together give the quotient. Thus, the quotient in the laſt example is found to be $a-x+\frac{2x^2}{a+x}$. And $bb+ab$ divided by $b-a$ gives for the quotient $b+\frac{2ab}{b-a}$.

Note, The ſign ÷ placed between any two quantities, expreſſed the quotient of the former divided by the latter. Thus $\overline{a+b}\div\overline{a-x}$ is the quotient of $a+b$ divided by $a-x$.

CHAP. VI.

Of FRACTIONS.

§ 22. IN the laſt Chapter it was ſaid that the quotient of any quantity *a* divided by *b* is expreſſed by placing *a* above a ſmall line and *b* under it, thus, $\frac{a}{b}$. Theſe quotients are alſo called *Fractions*; and the dividend or quantity placed above the line is called the *Numerator* of the fraction, and the diviſor or quantity placed under the line is called the *Denominator*. Thus $\frac{2}{3}$ expreſſed the quotient of 2 divided by 3; and 2 is the numerator and 3 the denominator of the fraction.

§ 23. *If the numerator of a fraction is equal to the denominator, then the fraction is equal to unity. Thus* $\frac{a}{a}$ *and* $\frac{b}{b}$ *are equal to unit. If the numerator is greater than the denominator, then the fraction is greater than unit.* In both theſe caſes, the fraction is called *improper*. But *if the numerator is leſs than the denominator, then the fraction is leſs than unit*, and is called *proper*. Thus $\frac{5}{3}$ is an improper fraction; but $\frac{3}{4}$ and $\frac{2}{3}$ are proper fractions. A *mixt* quantity is that whereof one part is an *integer* and the other a *fraction*. As $3\frac{4}{5}$ and $5\frac{2}{3}$, and $a+\frac{a^2}{b}$.

PROBLEM I.

§ 24. *To reduce a* MIXT *quantity to an* IMPROPER FRACTION.

Rule. *Multiply the part that is an integer by the denominator of the fractional part; and to the product add the numerator; under their ſum place the former denominator.*

Thus $2\frac{3}{5}$ reduced to an improper fraction, gives $\frac{13}{5}$; $a+\frac{a^2}{b}=\frac{ab+a^2}{b}$; and $a-x+\frac{a^2-ax}{x}=\frac{a^2-x^2}{x}$.

PROBLEM II.

§ 25. *To reduce an* IMPROPER *fraction to a* MIXT QUANTITY.

Rule. *Divide the numerator of the fraction by the denominator, and the quotient ſhall give the integral part; the remainder ſet over the denominator ſhall be the fractional part.*

Thus $\frac{12}{5}=2\frac{2}{5}$; $\frac{ab+a^2}{b}=a+\frac{a^2}{b}$; $\frac{ax+2xx}{a+x}=x+\frac{x^2}{a+x}$; $\frac{aa+xx}{a-x}=a+x+\frac{2xx}{a-x}$.

PROBLEM III.

§ 26. *To reduce fractions of different denominations to the fractions of equal value that ſhall have the ſame denominator.*

Rule. *Multiply each numerator, ſeparately taken, into all the denominators but its own, and the products ſhall give the new numerators. Then multiply all the denominators into one another, and the product ſhall give the common denominator.*

Thus the fractions $\frac{a}{b}$, $\frac{b}{c}$, $\frac{c}{d}$, are reſpectively equal to theſe fractions $\frac{acd}{bcd}$, $\frac{bbd}{bcd}$, $\frac{ccb}{bcd}$, which have the ſame denominator *bcd*. And the fractions $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, are reſpectively equal to theſe $\frac{40}{60}$, $\frac{45}{60}$, $\frac{48}{60}$.

PROBLEM IV.

§ 27. *To* ADD *and* SUBTRACT *Fractions.*

Rule. *Reduce them to a common denominator, and add or ſubtract the numerators, the ſum or difference ſet over the common denominator, is the ſum or remainder required.*

Thus $\frac{a}{b}+\frac{c}{d}+\frac{d}{e}=\frac{ade+bce+d^2b}{bde}$; $\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$; $\frac{2}{3}+\frac{3}{4}=\frac{8+9}{12}=\frac{17}{12}$ $=1\frac{5}{12}$; $\frac{3}{4}-\frac{2}{3}=\frac{9-8}{12}=\frac{1}{12}$;

$$\frac{4}{5}-\frac{3}{4}=\frac{16-15}{20}=\frac{1}{20};$$

$$\frac{x}{2}-\frac{x}{3}=\frac{3x-2x}{6}=\frac{x}{6}.$$

PROBLEM V.

§ 28. *To* MULTIPLY *Fractions.*

Rule. *Multiply their numerators one into another to obtain the numerator of the product; and their denominators multiplied into one another ſhall give the denominator of the product.*

Thus $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$; $\frac{2}{3}\times\frac{4}{5}=\frac{8}{15}$; and $\frac{a+b}{c}\times\frac{a-b}{d}=\frac{a^2-b^2}{cd}$.

If a mixt quantity is to be multiplied, firſt reduce it to the form of a fraction (by *Prob.* 1.) And if an integer is to be multiplied by a fraction, you may reduce it to the form of a fraction by placing unit under it.

EXAMPLES.

$5\frac{2}{3}\times\frac{3}{4}=\frac{17}{3}\times\frac{3}{4}=\frac{51}{12}$; $9\times\frac{2}{3}=\frac{9}{1}\times\frac{2}{3}$ $=$

$$= \frac{18}{3} = 6 : \overline{b + \frac{bx}{a}} \times \frac{a}{x} = \frac{ba + bx}{a} \times \frac{a}{x} = \frac{a^2b + abx}{ax} = \frac{ab + bx}{x}.$$

PROBLEM VI.

§ 29. *To* DIVIDE *Fractions.*

Rule. *Multiply the numerator of the dividend by the denominator of the diviſor, their product ſhall give the numerator of the quotient. Then multiply the denominator of the dividend by the numerator of the diviſor, and their product ſhall give the denominator.*

Thus $\frac{4}{5})\frac{2}{3}(\frac{10}{12}$; $\frac{3}{7})\frac{5}{8}(\frac{35}{24}$; $\frac{c}{d})\frac{a}{b}(\frac{ad}{cb}$;

$$\frac{a+b}{a-b})\frac{a-b}{a}(\frac{a^2-2ab+b^2}{a^2+ab}.$$

§ 30. Theſe laſt four Rules are eaſily demonſtrated from the definition of a fraction.

1. It is obvious that the fractions $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, are reſpectively equal to $\frac{adf}{bdf}$, $\frac{cbf}{dbf}$, $\frac{ebd}{fbd}$, ſince if you divide adf by bdf, the quotient will be the ſame as of a divided by b; and cbf divided by dbf gives the ſame quotient as c divided by d; and ebd divided by fbd the ſame quotient as e divided by f.

2. Fractions reduced to the ſame denominator are added by adding their numerators and

ſubſcribing the common denominator. I ſay $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$. For call $\frac{a}{b}=m$, and $\frac{c}{b}=n$, and it will be $a=mb$, $c=nb$, and $mb+nb=a+c$, and $m+n=\frac{a+c}{b}$; that is, $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$. After the ſame manner, $\frac{a}{b}-\frac{c}{b}=m-n=\frac{a-c}{b}$.

3. I ſay $\frac{a}{b}\times\frac{c}{d}\;(=m\times n)=\frac{ac}{bd}$; for $bm=a$, $dn=c$; and $bdmn=ac$, and $mn=\frac{ac}{bd}$; that is, $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$.

4. I ſay $\frac{a}{b}$ divided by $\frac{c}{d}$, or $\frac{m}{n}$, gives $\frac{ad}{cb}$; for $mb=a$, and $mbd=ad$; $nd=c$, and $nbd=cb$; therefore $\frac{mbd}{nbd}=\frac{ad}{cb}$; that is, $\frac{m}{n}=\frac{ad}{cb}$.

PROBLEM VII.

§ 31. *To find the greateſt common Meaſure of two numbers*; that is, the greateſt number that can divide them both without a remainder.

Rule. *Firſt divide the greater number by the leſſer, and if there is no remainder the leſſer number is the greateſt common diviſor required.*

If there is a remainder, divide your laſt diviſor by it; and thus proceed continually dividing the laſt diviſor by its remainder, till there is no remainder left, and then the laſt diviſor is the greateſt common Meaſure required.

Thus the greateſt common meaſure of 45 and 63 is 9; and the greateſt common meaſure of 256 and 48 is 16.

$$
\begin{array}{l}
45)\,63\,(1 \\
\quad \underline{45} \\
\quad 18)\,45\,(2 \\
\qquad\;\; \underline{36} \\
\qquad\;\;\; 9)\,18\,(2 \\
\qquad\qquad\; \underline{18} \\
\qquad\qquad\;\; 0
\end{array}
\qquad\qquad
\begin{array}{l}
48)\,256\,(5 \\
\qquad \underline{240} \\
\qquad\; 16)\,48\,(3 \\
\qquad\qquad\;\; \underline{48} \\
\qquad\qquad\quad 0
\end{array}
$$

§ 32. Much after the ſame manner the greateſt common meaſure of algebraic quantities is diſcovered; only *the remainders that ariſe in the operation are to be divided by thir ſimple diviſors*, and *the quantities are always to be ranged according to the dimenſions of the ſame letter.*

Thus to find the greateſt common meaſure of a^2-b^2 and $a^2-2ab+b^2$;

$$
\begin{array}{l}
a^2-b^2)\,a^2-2ab+b^2\,(1 \\
\qquad\qquad\; \underline{a^2-b^2\quad\;\;} \\
\qquad\qquad\qquad -2ab+2b^2 \text{ Remainder,}
\end{array}
$$

which divided by $-2b$ is reduced to

$$\begin{array}{l} a-b)\; a^2-b^2\; (a+b \\ \qquad\;\; a^2-b^2 \\ \hline \qquad\;\; 0 \quad\; 0 \end{array}$$

Therefore $a-b$ is the greateſt common meaſure required.

The ground of this operation is, That any quantity that meaſures the diviſor and the remainder (if there is any) muſt alſo meaſure the dividend; becauſe the dividend is equal to the ſum of the diviſor multiplied into the quotient, and of the remainder added together*. Thus in the laſt example, $a-b$ meaſures the diviſor a^2-b^2, and the remainder $-2ab+2b$; it muſt therefore likewiſe meaſure their ſum $a^2-2ab+b^2$. You muſt obſerve in this operation to make that the dividend which has the higheſt powers of the letter, according to which the quantities are ranged.

PROBLEM VIII.

§ 33. *To reduce any Fraction to its loweſt terms.*

Rule. *Find the greateſt common meaſure of the numerator and denominator; divide them by that common meaſure and place the quotients in their room, and you ſhall have a fraction equivalent*

---

* See Chap. XIV.

to the given fraction expreſſed in the leaſt terms.

$$\text{Thus } \frac{25bc}{25bc}\Big)\frac{75abc}{125bcx} = \frac{3a}{5x};\quad \frac{156aa+156ab}{572aa-572ab} = \frac{3a+3b}{11a-11b};\quad \frac{a^2-b^2}{a^2-2ab+b^2} = \frac{a+b}{a-b};$$

$$\frac{a^3-b^2a}{a^2+2ab+b^2} = \frac{a^2-ba}{a+b};\quad \frac{a^4-b^4}{a^5-a^3b^2} = \frac{a^2+b^2}{a^3}.$$

When unit is the greateſt common meaſure of the numbers and quantities, then the fraction is already in its loweſt terms. Thus $\frac{3ab}{5dc}$ can not be reduced lower.

And numbers whoſe greateſt common meaſure is unit, are ſaid to be *prime* to one another.

§ 34. If it is required to reduce a given fraction to a fraction equal to it that ſhall have a given denominator, you muſt *multiply the numerator by the given denominator, and divide the product by the former denominator, the quotient ſet over the given denominator is the fraction required.* Thus $\frac{a}{b}$ being given, and it being required to reduce it to an equal fraction whoſe denominator ſhall be c; find the quotient of ac divided by b, and it ſhall be the numerator of the fraction required.

If a Vulgar fraction is to be reduced to a decimal (that is, a fraction whoſe denominator is 10, or any of its powers) *annex as many cyphers as you pleaſe to the numerator, and then divide it by the denominator, the quotient ſhall give a decimal equal to the Vulgar fraction propoſed.*

$$\text{Thus } \frac{2}{3} = .66666, \&c.; \quad \frac{3}{5} = .6; \quad \frac{2}{7} = .2857142, \&c.$$

§ 35. Theſe fractions are added and ſubtracted like whole numbers; only care muſt be taken to *ſet ſimilar places above one another*, as units above units, and tenths above tenths, *&c.* They are multiplied and divided as integer numbers; only *there muſt be as many decimal places in the product as in both the multiplicand and multiplier*; and *in the quotient as many as there are in the dividend more than in the diviſor.* And in diviſion the quotient may be continued to any degree of exactneſs you pleaſe, by adding cyphers to the dividend. The ground of theſe operations is eaſily underſtood from the general rules for adding, multiplying, and dividing fractions.

CHAP.

CHAP. VII.

Of the INVOLUTION of Quantities.

§ 36. THE productſs ariſing from the continual multiplication of the ſame quantity were called (in *Chap.* 4.) the *powers* of that quantity. Thus a, a^2, a^3, a^4, &c. are the powers of a; and ab, a^2b^2, a^3b^3, a^4b^4, &c. are the powers of ab. In the ſame Chapter, the rule for the multiplication of powers of the ſame quantity is to "Add the exponents and make their ſum the exponent of the product." Thus $a^4 \times a^5 = a^9$; and $a^3b^3 \times a^6b^2 = a^9b^5$. In *Chap.* 5. you have the rule for dividing powers of the ſame quantity, which is, "To ſubtract the exponents and make the difference the exponent of the quotient."

Thus $\frac{a^6}{a^4} = a^{6-4} = a^2$; and $\frac{a^5b^3}{a^4b} = a^{5-4}b^{3-1} = ab^2$.

§ 37. *If you divide a leſſer power by a greater, the exponent of the quotient muſt,* by this Rule, *be negative.* Thus $\frac{a^4}{a^6} = a^{4-6} = a^{-2}$. But $\frac{a^4}{a^6} = \frac{1}{a^2}$; and hence $\frac{1}{a^2}$ is expreſſed alſo by a^2 with a negative exponent.

It is alſo obvious that $\frac{a}{a} = a^{1-1} = a^0$; but $\frac{a}{a} = 1$, and therefore $a^0 = 1$. After the ſame manner $\frac{1}{a} = \frac{a^0}{a} = a^{0-1} = a^{-1}$; $\frac{1}{aa} = \frac{a^0}{a^2} = a^{0-2} = a^{-2}$; $\frac{1}{aaa} = a^{0-3} = a^{-3}$; ſo that the quantities a, 1, $\frac{1}{a}$, $\frac{1}{a^2}$, $\frac{1}{a^3}$, $\frac{1}{a^4}$, &c. may be expreſſed thus, a^1, a^0, a^{-1}, a^{-2}, a^{-3}, a^{-4}, &c. Thoſe are called the *negative* powers of a which have negative exponents; but they are at the ſame time *poſitive* powers of $\frac{1}{a}$ or a^{-1}.

§ 38. *Negative powers (as well as poſitive) are multiplied by adding, and divided by ſubtracting their exponents.* Thus the product of a^{-2} (or $\frac{1}{a^2}$) multiplied by a^{-3} (or $\frac{1}{a^3}$) is $a^{-2-3} = a^{-5}$ (or $\frac{1}{a^5}$); alſo $a^{-6} \times a^4 = a^{-6+4} = a^{-2}$ (or $\frac{1}{a^2}$); and $a^{-3} \times a^3 = a^0 = 1$. And, in general, *any poſitive power of a multiplied by a negative power of a of an equal exponent gives* UNIT *for the product*; for the poſitive and negative deſtroy each other, and the product gives a^0, which is equal to unit.

Likewiſe $\frac{a^{-5}}{a^{-2}} = a^{-5+2} = a^{-3} = \frac{1}{a^3}$; and $\frac{a^{-2}}{a^{-5}} = a^{-2+5} = a^3$. But alſo, $\frac{a^{-2}}{a^{-5}} = \frac{a^{-2}}{a^{-2} \times a^{-3}} = \frac{1}{a^{-3}}$; therefore $\frac{1}{a^{-3}} = a^3$: And, in general, "any quantity placed in the denominator of a fraction may be tranſpoſed to the numerator, if the ſign of its exponent be changed." Thus $\frac{1}{a^3} = a^{-3}$, and $\frac{1}{a^{-3}} = a^3$.

§ 39. The quantity a^m expreſſes any power of a in general; the exponent (m) being undetermined; and a^{-m} expreſſes $\frac{1}{a^m}$, or a negative power of a of an equal exponent: and $a^m \times a^{-m} = a^{m-m} = a^0 = 1$ is their product. a^n expreſſes any other power of a; $a^m \times a^n = a^{m+n}$ is the product of the powers a^m and a^n, and a^{m-n} is their quotient.

§ 40. To raiſe any ſimple quantity to its ſecond, third, or fourth power, is to add its exponent twice, thrice, or four times to itſelf; therefore the ſecond power of any quantity is had by doubling its exponent, and the third by trebling its exponent; and, in general, *the power expreſſed by m of any quantity is had by multiplying the exponent by m*, as is obvious from the multiplication of powers. Thus the ſecond power or ſquare of a is $a^{2\times 1} = a^2$; its third power or cube is $a^{3\times 1}=a^3$; and the *m*th power of *a* is $a^{m\times 1}=a^m$. Alſo, the ſquare of a^4 is $a^{2\times 4}=a^8$; the cube of a^4 is $a^{3\times 4}=a^{12}$; and the *m*th power of a^4 is $a^{4\times m}$. The ſquare of *abc* is $a^2b^2c^2$, the cube is $a^3b^3c^3$, the *m*th power $a^mb^mc^m$.

§ 41. The raiſing of quantities to any power is called *Involution*; and *any ſimple quantity is involved by multiplying the exponent by that of the power required*, as in the preceding Examples. *The coefficient muſt alſo be raiſed to the ſame power* by continual multiplication of itſelf by itſelf, as often as unit is contained in the exponent of the power required. Thus the cube of $3ab$ is $3\times 3\times 3\times a^3b^3=27a^3b^3$.

As to the Signs, *When the quantity to be involved is poſitive, it is obvious that all its powers muſt be poſitive.* And *when the quantity to be involved is negative, yet all its powers whoſe exponents are even numbers muſt be poſitive,* for any number of multiplications of a negative, if the number is even, gives a poſitive; ſince $-\times-=+$, therefore $-\times-\times-\times-=+\times+=+$; and $-\times-\times-\times-\times-\times-=+\times+\times+=+$.

The power then only can be negative when its exponent is an odd number, though the quantity to be involved be negative. The powers of $-a$ are $-a$, $+a^2$, $-a^3$, $+a^4$, $-a^5$,

&c. Thoſe whoſe exponents are 2, 4, 6, *&c.* are poſitive; but thoſe whoſe exponents are 1, 3, 5, *&c.* are negative.

§ 42. The involution of *compound* quantities is a more difficult operation. The powers of any *binomial* $a+b$ are found by continual multiplication of it by itſelf as follows.

$$
\begin{array}{l}
a+b=\text{Root.}\\
\times\ \underline{a+b}\\
a^2+ab\\
+ab+b^2\\
\hline
a^2+2ab+b^2=\text{the Square or 2d Power.}\\
\times\ \underline{a+b}\\
a^3+2a^2b+ab^2\\
+a^2b+2ab^2+b^3\\
\hline
a^3+3a^2b+3ab^2+b^3=\text{Cube or 3d Power.}\\
\times\ \underline{a+b}\\
a^4+3a^3b+3a^2b^2+ab^3\\
+a^3b+3a^2b^2+3ab^3+b^4\\
\hline
a^4+4a^3b+6a^2b^2+4ab^3+b^4=\text{Biquadrate or 4th Power.}\\
\times\ \underline{a+b}\\
a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4\\
+a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5\\
\hline
a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5=\text{5th Power.}\\
\times\ \underline{a+b}\\
a^6+5a^5b+10a^4b^2+10a^3b^3+5a^2b^4+ab^5\\
+a^5b+5a^4b^2+10a^3b^3+10a^2b^4+5ab^5+b^6\\
\hline
a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6=\text{6th Power, \&c.}
\end{array}
$$

§ 43. If the powers of $a-b$ are required, they will be found the ſame as the preceding, only the terms in which the exponent of b is an odd number will be found negative; "becauſe an odd number of multiplications of a negative produces a negative." Thus the cube of $a-b$ will be found to be $a^3-3a^2b+3ab^2-b^3$: where the 2d and 4th terms are negative, the exponent of b being an odd number in theſe terms. In general, "The terms of any power of $a-b$ are poſitive and negative by turns."

§ 44. It is to be obſerved, That "in the firſt term of any power of $a \mp b$, the quantity a has the exponent of the power required, that in the following terms, the exponent of a decreaſe gradually by the ſame difference (*viz.* unit) and that in the laſt terms it is never found. The powers of b are in the contrary order; it is not found in the firſt term, but its exponent in the ſecond term is unit, in the third term its exponent is 2; and thus its exponent increaſes, till in the laſt term it becomes equal to the exponent of the power required."

As the exponents of a thus decreaſe, and at the ſame time thoſe of b increaſe, "the ſum of their exponents is always the ſame, and is equal to the exponent of the power required." Thus in the 6th power of $a+b$, viz. $a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$, the exponents of a decreaſe in this order, 6, 5, 4, 3, 2, 1, 0; and thoſe of b increaſe in the contrary order, 0, 1, 2, 3, 4, 5, 6. And the ſum of their exponents in any term is always 6.

§ 45. To find their coefficient of any term, the coefficient of the precedent term being known; you are to "divide the coefficient of the preceding term by the exponent of b in the given term, and to multiply the quotient by the exponent of a in the ſame term, increaſed by unit. Thus to find the coefficients of the terms of the 6th power of $a+b$, you find the terms are

$$a^6,\ a^5b,\ a^4b^2,\ a^3b^3,\ a^2b^4,\ ab^5,\ b^6;$$

and you know the coefficient of the firſt term is unit; therefore, according to the rule, the coefficient of the 2d term will be $\frac{1}{1}\times\overline{5+1}=6$; that of the 3d term will be $\frac{6}{2}\times\overline{4+1}=3\times5=15$; that of the 4th term will be $\frac{15}{3}\times\overline{3+1}=5\times4=20$; and thoſe of the following will be 15, 6, 1, agreeable to the preceding Table.

§ 46. In general, if $a+b$ is to be raiſed to any power m, the terms, without their coefficients, will be, a^m, $a^{m-1}b$, $a^{m-2}b^2$, $a^{m-3}b^3$, $a^{m-4}b^4$, $a^{m-5}b^5$, *&c.* continued till the exponent of b becomes equal to m.

The coefficients of the reſpective terms, according to the laſt rule, will be $1, m, m \times \frac{m-1}{2}, m \times \frac{m-1}{2} \times \frac{m-2}{3}, m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}, m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$, *&c.* continued until you have one coefficient more than there are units in *m*.

It follows therefore by theſe laſt rules, that $\overline{a+b}^m = a^m + ma^{m-1}b + m \times \frac{m-1}{2} \times a^{m-2}b^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times a^{m-3}b^3 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times a^{m-4}b^4 +$, *&c.* which is the general *Theorem* for raiſing a quantity conſiſting of two terms to any power *m*.

§ 47. If a quantity conſiſting of three, or more terms is to be involved, "you may diſtinguiſh it into two parts, conſidering it as a binomial, and raiſe it to any power by the preceding rules; and then by the ſame rules you may ſubſtitute inſtead of the powers of theſe compound parts their values."

Thus $\overline{a+b+c}^2 = \overline{\overline{a+b}+c}^2 = \overline{a+b}^2 + 2c \times \overline{a+b} + c^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2$.

And $\overline{a+b+c}^3 = \overline{a+b}^3 + 3c \times \overline{a+b}^2 + 3c^2 \times \overline{a+b} + c^3 = a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$.

In theſe examples, $a + b + c$ is conſidered as compoſed of the compound part $a + b$ and the ſimple part c; and then the powers of $a + b$ are formed by the preceding rules, and ſubſtituted for $\overline{a+b}^3$ and $\overline{a+b}^2$.

* * *

CHAP. VIII.

Of EVOLUTION.

§ 48. THE reverſe of Involution, or the reſolving of powers into their roots is called *Evolution*. The roots of ſingle quantities are eaſily extracted *by dividing their exponents by the number that denominates the root required.* Thus the ſquare root of a^8 is $a^{\frac{8}{2}} = a^4$; and the ſquare root of $a^4b^8c^2$ is a^2b^4c. The cube root of a^6b^3 is $a^{\frac{6}{3}}b^{\frac{3}{3}} = a^2b$; and the cube root of $x^9y^6z^{12}$ is $x^3y^2z^4$. The ground of this rule is obvious from the rule for Involution. The powers of any root are found by multiplying its exponent by the index that denominates the power; and therefore, when any power is given, the root muſt be found by dividing the exponent of the given power by the number that denominates the kind of root that is required.

§ 49. It appears from what was ſaid of Involution, that "*any power that has a poſitive ſign may have either a poſitive or negative root, if the root is denominated by any even number.*" Thus the ſquare root of $+a^2$ may be $+a$ or $-a$, becauſe $+a \times +a$ or $-a \times -a$ gives $+a^2$ for the product.

But if a power have a negative ſign, "*no root of it denominated by an even number can be aſſigned,*" ſince there is no quantity that multiplied into itſelf an even number of times can give a negative product. Thus the ſquare root of $-a^2$ cannot be aſſigned, and is what we call an "*impoſſible* or *imaginary* quantity.

But if the root to be extracted is denominated by an *odd* number, *then ſhall the ſign of the root be the ſame as the ſign of the given number whoſe root is required.* Thus the cube root of $-a^3$ is $-a$, and the cube root of a^6b^3 is $-a^2b$.

§ 50. If the number that denominates the root required is a diviſor of the exponent of the given power, then ſhall the root be only a "*lower power of the ſame quantity.*" As the cube root of a^{12} is a^4, the number 3 that denominates the cube root being a diviſor of 12.

But if the number that denominates what ſort of root is required is not a diviſor of the exponent of the given power, "*then the root required ſhall have a fraction for its exponent.*"

Thus the ſquare root of a^3 is $a^{\frac{3}{2}}$; the cube root of a^5 is $a^{\frac{5}{3}}$, and the ſquare root of a itſelf is $a^{\frac{1}{2}}$

Theſe powers that have fractional exponents are called "*Imperfect powers* or *ſurds*;" and are otherwiſe expreſſed by placing the given power within the radical ſign $\sqrt{}$, and placing above the radical ſign the number that denominates what kind of root is required. Thus $a^{\frac{3}{2}} = \sqrt[2]{a^3}$; $a^{\frac{5}{3}} = \sqrt[3]{a^5}$; and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$. In numbers the ſquare root of 2 is expreſſed by $\sqrt[2]{2}$, and the cube root of 4 by $\sqrt[3]{4}$.

§ 51. Theſe imperfect powers or ſurds are "*multiplied and divided*, as other powers, *by adding and ſubtracting their exponents.*" Thus $a^{\frac{1}{2}} \times a^{\frac{5}{2}} = a^{\frac{6}{2}} = a^3$; $a^{\frac{2}{3}} \times a^{\frac{3}{4}} = a^{\frac{2}{3}+\frac{3}{4}} = a^{\frac{17}{12}} = \sqrt[12]{a^{17}}$; and $\frac{a^{\frac{7}{2}}}{a^{\frac{3}{2}}} = a^{\frac{7-3}{2}} = a^{\frac{4}{2}} = a^2$.

They are *involved* likewiſe and *evolved* after the ſame manner as perfect powers. Thus the ſquare of $a^{\frac{3}{2}}$ is $a^{\frac{2\times3}{2}} = a^3$; the cube of $a^{\frac{2}{3}}$ is $a^{3\times\frac{2}{3}} = a^{\frac{6}{3}}$. The ſquare root of $a^{\frac{2}{3}}$ is $a^{\frac{2}{3\times2}} = a^{\frac{1}{3}}$, the cube root of $a^{\frac{3}{4}}$ is $a^{\frac{1}{4}}$. But we ſhall have occaſion to treat more fully of Surds hereafter.

§ 52. The ſquare root of any compound quantity, as $a^2 + 2ab + b^2$ is diſcovered after this manner. "*Firſt, take care to diſpoſe the terms according to the dimenſions of the alphabet, as in diviſion; then find the ſquare root of the firſt term aa, which gives a for the firſt member of the root. Then ſubtract its ſquare from the propoſed quantity, and divide the firſt term of the remainder* ($2ab+b^2$) *by the double of that member,* viz. $2a$, *and the quotient b is the ſecond member of the root. Add this ſecond member to the double of the firſt, and multiply their ſum* ($2a+b$) *by the ſecond member b, and ſubtract the product* ($2ab+b^2$) *from the foreſaid remainder* ($2ab+b^2$) *and if nothing remains, then the ſquare root is obtained*; and in this example it is found to be $a+b$.

The manner of the operation is thus,

$$\begin{array}{rl} & a^2+2ab+b^2\,(a+b \\ & a^2 \\ \hline 2a+b\big) & 2ab+b^2 \\ \times b\big) & 2ab+b^2 \\ \hline & 0 \quad 0 \end{array}$$

But if there had been a remainder, you muſt have divided it by the double of the ſum of the two parts already found, and the quotient would have given the third member of the root.

Thus if the quantity propoſed had been $a^2+2ab+2ac+b^2+2bc+c^2$, after proceeding as above you would have found the remainder

$2ac+2bc+c^2$, which divided by $2a+2b$ gives c to be annexed to $a+b$ as the 3d member of the root. Then adding c to $2a+2b$ and multiplying their ſum $2a+2b+c$ by c, ſubtract the product $2ac+2bc+c^2$ from the foreſaid remainder; and ſince nothing now remains, you conclude that $a+b+c$ is the ſquare root required.

The operation is thus;

$$\begin{array}{rl}
 & a^2+2ab+2ac+b^2+2bc+c^2\ (a+b+c \\
 & a^2 \\ \hline
2a+b\ \Big) & 2ab+2ac+b^2+2bc+c^2 \\
\times b\ \Big) & 2ab \qquad\quad +b^2 \\ \hline
2a+2b+c\ \Big) & 2ac+2bc+c^2 \\
\times c\ \Big) & 2ac+2bc+c^2 \\ \hline
 & 0 \qquad 0 \qquad 0
\end{array}$$

Another Example.

$$\begin{array}{rl}
 & xx-ax+\tfrac{1}{4}aa\ (x-\tfrac{1}{2}a \\
 & xx \\ \hline
2x-\tfrac{1}{2}a\ \Big) & -ax+\tfrac{1}{4}aa \\
\times -\tfrac{1}{2}a\ \Big) & -ax+\tfrac{1}{4}aa \\ \hline
 & 0 \qquad 0
\end{array}$$

The ſquare root of any number is found out after the ſame manner. If it is a number under 100, its neareſt ſquare root is found by the following Table; by which alſo its cube root is found if it be under 1000, and its biquadrate if it be under 10000.

The Root	1	2	3	4	5	6	7	8	9
Square	1	4	9	16	25	36	49	64	81
Cube	1	8	27	64	125	216	343	512	729
Biquad.	1	16	81	256	625	1296	2401	4096	6561

But if it is a number above 100, then its ſquare root will conſiſt of two or more figures, which muſt be found by different operations by the following

RULE.

§ 53. *Place a point above the number that is in the place of units, paſs the place of tens, and place again a point over that of hundreds, and go on towards the left hand, placing a point over every 2d figure; and by theſe points the number will be diſtinguiſhed into as many parts as there are figures in the root. Then find the ſquare root of the firſt part, and it will give the firſt figure of the root; ſubtract its ſquare from that part, and annex the ſecond part of the given number to the remainder. Then divide this new number (neglecting its laſt figure) by the double of the firſt figure of the root, annex the quotient to that double, and multiply the number thence ariſing by the ſaid quotient, and if the product is leſs than your dividend, or equal to it, that quotient ſhall be the ſecond figure of the root. But if the product is greater than the dividend, you muſt take a leſs number for the ſecond figure*

figure of the root than that quotient. Much after the ſame manner may the other figures of the quotient be found, if there are more points than two places over the given number.

To find the ſquare root of 99856, I firſt point it thus $\dot{9}9\dot{8}5\dot{6}$, then I find the ſquare root of 9 to be 3, which therefore is the firſt figure of the root; I ſubtract 9, the ſquare of 3, from 9, and to the remainder I annex the ſecond part 98, and I divide (neglecting the laſt figure 8) by the double of 3, or 6, and I place the quotient after 6, and then multiply 61 by 1, and ſubtract the product 61 from 98. Then to the remainder (37) I annex the laſt part of the propoſed number (56) and dividing 3756 (neglecting the laſt figure 6) by the double of 31, that is by 62, I place the quotient after, and multiplying 626 by the quotient 6, I find the product to be 3756, which ſubtracted from the dividend and leaving no remainder, the exact root muſt be 316.

EXAMPLES.

$$\begin{array}{rl}
 & \dot{9}9\dot{8}5\dot{6}\,(316 \\
 & \underline{9} \\
61 \big) & 98 \\
\times 1 \big) & \underline{61} \\
626 \big) & 3756 \\
\times 6 \big) & \underline{3756} \\
 & 0
\end{array}$$

$$\begin{array}{rl}
 & 2\dot{7}3\dot{9}4\dot{7}5\dot{6}(5234 \\
 & \underline{25} \\
102\big) & 239 \\
\times 2\big) & \underline{204} \\
1043\big) & 3547 \\
\times 3\big) & \underline{3129} \\
10464\big) & 41856 \\
\times 4\big) & \underline{41856} \\
 & 0
\end{array}
\qquad\Bigg|\qquad
\begin{array}{rl}
 & \dot{5}2\dot{9}(23 \\
 & \underline{4} \\
43\big) & 129 \\
\times 3\big) & \underline{129} \\
 & 0
\end{array}$$

§ 54. In general, to extract any root out of any given quantity, "*Firſt range that quantity according to the dimenſions of its letters, and extract the ſaid root out of the firſt term, and that ſhall be the firſt member of the firſt root required. Then raiſe this root to a dimenſion lower by unit than the number that denominates the root required, and multiply the power that ariſes by that number itſelf; divide the ſecond term of the given quantity by the product, and the quotient ſhall give the ſecond member of the root required.*"

Thus to extract the root of the 5th power out of $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$, I find that the root of the 5th power out of a^5 gives a, which I raiſe to the 4th power, and multiplying by 5, the product is $5a^4$; then dividing the ſecond term of the given quantity $5a^4b$ by $5a^4$, I find b to be the ſecond member;

and raiſing $a+b$ to the 5th power and ſubtracting it, there being no remainder, I conclude that $a+b$ is the root required. If the root has three members, the third is found after the ſame manner from the firſt two conſidered as one member, as the ſecond member was found from the firſt; which may be eaſily underſtood from what was ſaid of extracting the ſquare root.

§ 55. In extracting roots it will often happen that the exact root cannot be found in finite terms; thus the ſquare root of a^2+x^2 is found to be

$$a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}+,\ \&c.$$

The operation is thus;

$$\begin{array}{rl}
 & a^2+x^2\left(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-,\ \&c.\right. \\
 & \underline{a^2} \\
2a+\frac{x^2}{2a}\Big) & *+x^2 \\
\times\frac{x^2}{2a}\Big) & =x^2+\frac{x^4}{4a^2} \\
\hline
2a+\frac{x^2}{a}-\frac{x^4}{8a^3}\Big) & -\frac{x^4}{4a^2} \\
\times-\frac{x^4}{8a^3}\Big) & =-\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6} \\
\hline
 & +\frac{x^6}{8a^4}-\frac{x^8}{64a^6} \\
 & \&c.
\end{array}$$

After the ſame manner, the cube root of a^3+x^3 will be found to be

$$a+\frac{x^3}{3a^2}-\frac{x^6}{9a^5}+\frac{5x^9}{81a^8}-\frac{10x^{12}}{243a^{11}}+,\ \&c.$$

§ 56. "The general Theorem which we gave for the Involution of binomials will ſerve alſo for their Evolution;" becauſe to extract any root of a given quantity is the ſame thing as to raiſe that quantity to a power whoſe exponent is a fraction that has unity for its numerator, and the number that expreſſes what kind of root is to be extracted for its denominator. Thus, to extract the ſquare root of $a+b$, is to raiſe $a+b$ to a power whoſe exponent is $\frac{1}{2}$: Now ſince $\overline{a+b}|^{m}=a^m+m\times a^{m-1}b+m\times\frac{m-1}{2}a^{m-2}b^2+m\times\frac{m-1}{2}\times\frac{m-2}{3}\times a^{m-3}b^3$, &c.

Suppoſing $m=\frac{1}{2}$, you will find

$\overline{a+b}|^{\frac{1}{2}}=a^{\frac{1}{2}}+\frac{1}{2}\times a^{-\frac{1}{2}}b+\frac{1}{2}\times-\frac{1}{4}\times a^{-\frac{3}{2}}b^2+\frac{1}{2}\times-\frac{1}{4}\times-\frac{1}{2}a^{-\frac{5}{2}}b^3$, &c. $=a^{\frac{1}{2}}+\frac{b}{2a^{\frac{1}{2}}}-\frac{b^2}{8a^{\frac{3}{2}}}+\frac{b^3}{16a^{\frac{5}{2}}}-$, &c. And after this manner you will find that

$\overline{a^2+x^2}|^{\frac{1}{2}}=a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-$, &c. as before.

§ 57. The roots of numbers are to be extracted as thoſe of algebraic quantities. "*Place a point over the units, and then place points over every*

every third, fourth or fifth figure towards the left hand, according as it is the root of the cube, of the 4th or 5th power that is required; and if there be any decimals annexed to the number, point them after the ſame manner, proceeding from the place of units towards the right hand. By this means the number will be divided into ſo many periods as there are figures in the root required. Then enquire which is the greateſt cube, biquadrate, or 5th power in the firſt period, and the root of that power will give the firſt figure of the root required. Subtract the greateſt cube, biquadrate, or 5th power from the firſt period, and to the remainder annex the firſt figure of your ſecond period, which ſhall give your dividend.

Raiſe the firſt figure already found to a power leſs by unit than the power whoſe root is ſought, that is, to the 2d, 3d, or 4th power, according as it is the cube root, the root of the 4th, or the root of the 5th power that is required, and multiply that power by the index of the cube, 4th or 5th power, and divide the dividend by this product, ſo ſhall the quotient be the ſecond figure of the root required.

Raiſe the part already found of the root, to the power whoſe root is required, and if that power be found leſs than the two firſt periods of the given number, the ſecond figure of the root is right. But if it be found greater, you muſt diminiſh the ſecond figure of the root till that power *be found equal to or leſs than thoſe periods of the given number. Subtract it, and to the remainder annex the next period; and proceed till you have gone through the whole given number, finding the 3d figure by means of the two firſt, as you found the ſecond by the firſt; and afterwards finding the 4th figure (if there be a 4th period) after the ſame manner from the three firſt."*

Thus to find the cube root of 13824; point it $1\dot{3}82\dot{4}$; find the greateſt cube in 13, *viz.* 8, whoſe cube root 2 is the firſt figure of the root required. Subtract 8 from 13, and to the remainder 5 annex 8 the firſt figure of the ſecond period; divide 58 by triple the ſquare of 2, *viz.* 12. and the quotient is 4, which is the ſecond figure of the root required, ſince the cube of 24 gives 13824, the number propoſed. After the ſame manner the cube root of 13312053 is found to be 237.

OPERATION.

$1\dot{3}82\dot{4}$ (24
Subtr. 8=2×2×2

3×4=12) 58 (4
Subtract 24×24×24=13824

Rem. 0

$13312053\ (237$

$8 = 2\times2\times2$

$12)\ 53\ (4\ \text{or})\ 3$

Subtract $12167 = 23\times23\times23$

$3\times23\times23 = 1587)\ 11450\ (7$

Subtract $13312053 = 237\times237\times237$

Remain. 0

In extracting of roots, after you have gone through the number propoſed, if there is a remainder, you may continue the operation by adding periods of cyphers to that remainder, and find the true root in decimals to any degree of exactneſs.

CHAP. IX.

Of PROPORTION.

§ 58. WHEN quantities of the ſame kind are compared, it may be conſidered either how much the one is greater than the other, and what is their *difference*; or, it may be conſidered how many times the one is contained in the other; or, more generally, what is their *quotient*. The firſt relation of quantities is expreſſed by their *Arithmetical ratio*; the ſecond by their *Geometrical ratio*. That term whoſe ratio is enquired into is called the *antecedent*, and that with which it is compared is called the *conſequent*.

§ 59. When of four quantities the difference betwixt the firſt and ſecond is equal to the difference betwixt the third and fourth, thoſe quantities are called *Arithmetical proportionals*; as the numbers 3, 7, 12, 16. And the quantities, a, $a+b$, e, $e+b$. But quantities form a *ſeries* in arithmetical proportion, when they "*increaſe or decreaſe by the ſame conſtant difference.*" As theſe, a, $a+b$, $a+2b$, $a+3b$, $a+4b$, &c. x, $x-b$, $x-2b$, &c. or the numbers, 1, 2, 3, 4, 5, &c. and 10, 7, 4, 1, −2, −5, −8, &c.

§ 60. In four quantities *arithmetically proportional*, "*the ſum of the extremes is equal to the ſum of the mean terms.*" Thus a, $a+b$, e, $e+b$, are arithmetical proportionals, and the ſum of the extremes $(a+e+b)$ is equal to the ſum of the mean terms $(a+b+e)$. Hence, to find the fourth quantity arithmetically proportional to any three given quantities; "Add the ſecond and third, and from their ſum ſubtract the firſt term, the remainder ſhall give the fourth arithmetical proportional required."

§ 61. In a *ſeries* of arithmetical proportionals "*the ſum of the firſt and laſt term is equal to the ſum of any two terms equally diſtant from the extremes.*" If the firſt terms are a, $a+b$, $a+2b$, &c. and the laſt term x, the laſt term but one will be $x-b$, the laſt but two $x-2b$, the laſt but three $x-3b$, &c. So that the firſt half of the terms, having thoſe that are equally diſtant from the laſt term ſet under them, will ſtand thus;

$$\begin{array}{l} a,\ a+b,\ a+2b,\ a+3b,\ a+4b, \\ x,\ x-b,\ x-2b,\ x-3b,\ x-4b, \end{array} \&c.$$
$$\overline{a+x,\ a+x,\ a+x,\ a+x,\ a+x,\ \&c.}$$

And it is plain that if each term be added to the term above it, the ſum will be $a+x$ equal to the ſum of the firſt term a and the laſt term x. From which it is plain, that "*the ſum of all the terms of an arithmetical progreſſion is equal to the ſum of the firſt and laſt taken half as often as there are terms,*" that is, the ſum of an arithmetical progreſſion is equal to the ſum of the firſt and laſt terms multiplied by half the number of terms. Thus in the preceding ſeries, if n be the number of terms, the ſum of all the terms will be $\overline{a+x}\times\frac{n}{2}$.

§ 62. The common difference of the terms being b, and b not being found in the firſt term, it is plain that " its coefficient in any term will be equal to the number of terms that precede that term." Therefore in the laſt term x you muſt have $\overline{n-1}\times b$, ſo that x muſt be equal to $a+\overline{n-1}\times b$. And the ſum of all the terms being $\overline{a+x}\times\frac{n}{2}$, *it will alſo be equal to* $\frac{2an+n^2b-nb}{2}$, *or to* $a+\frac{nb-b}{2}\times n$. Thus for example, the ſeries $1+2+3+4+5$, &c. continued to a hundred, muſt be equal to $\frac{2\times100+10000-100}{2}=5050$.

§ 63. If a ſeries have (0) nothing for its firſt term, then "*its ſum ſhall be equal to half the product of the laſt term multiplied by the number of terms.*" For then, a being $=0$, the ſum of the terms, which is in general $\overline{a+x}\times\frac{n}{2}$, will in this caſe be $\frac{nx}{2}$. From which it is evident, that "the ſum of any number of arithmetical proportionals beginning from nothing, is equal to half the ſum of as many terms equal to the greateſt term.

Thus $0+1+2+3+4+5+6+7+8+9=\frac{9+9+9+9+9+9+9+9+9+9}{2}=\frac{10\times9}{2}=45$.

§ 64. If the four quantities the quotient of the *firſt* and *ſecond* be equal to the quotient of the *third* and *fourth*, then thoſe quantities are ſaid to be in *Geometrical proportion.*" Such are the numbers 2, 6, 4, 12; and the quantities *a*, *ar*, *b*, *br*; which are expreſſed after this manner;

$$2 : 6 :: 4 : 12.$$
$$a : ar :: b : br.$$

And you read them by ſaying, As 2 is to 6, ſo is 4 to 12; or as *a* is to *ar*, ſo is *b* to *br*.

In four quantities geometrically proportional, "*the product of the extremes is equal to the product of the middle terms.*" Thus $a \times br = ar \times b$. And, if it is required to find a fourth proportional to any three given quantities, "*multiply the ſecond by the third, and divide their product by the firſt, the quotient ſhall give the fourth proportional required.*" Thus, to find a fourth proportional to *a*, *ar*, and *b*, I multiply *ar* by *b*, and divide the product *arb* by the firſt term *a*, the quotient *br* is the fourth proportional required.

§ 65. In calculations it ſometimes requires a little care to place the terms in due order; for which you may obſerve the following Rule.

"*Firſt ſet down the quantity that is of the ſame kind with the quantity ſought, then conſider, from the nature of the queſtion, whether that which is given is greater or leſs than that which is ſought; if it is greater, then place the greateſt of the other two quantities on the left hand; but if it is leſs, place the leaſt of the other two quantities on the left hand, and the other on the*

the right," Then ſhall the terms be in due order; and you are to proceed according to the rule, multiplying the ſecond by the third, and dividing their product by the firſt.

EXAMPLE.

If 30 *men do any piece of work in* 12 *days, how many men ſhall do it in* 18 *days?*

Becauſe it is a number of men that is ſought, firſt ſet down 30, the number of men that is given: I eaſily ſee that the number that is given is greater than the number that is ſought, therefore I place 18 on the left hand, and 12 on the right; and find a fourth proportional to 18, 30, 12, *viz.* $\frac{30\times 12}{18}=20$.

§ 66. When a ſeries of quantities increaſe by one common multiplicator, or decreaſe by one common diviſor, they are ſaid to be in "*Geometrical proportion continued.*"

$$\text{As } a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5, \text{ \&c. or,}$$

$$a,\ \frac{a}{r},\ \frac{a}{r^2},\ \frac{a}{r^3},\ \frac{a}{r^4},\ \frac{a}{r^5}, \text{ \&c.}$$

The common multiplier or diviſor is called their "*common ratio.*"

In ſuch a ſeries, "*the product of the firſt and laſt is always equal to the product of the ſecond and laſt but one,* or *to the product of any two terms equally remote from the extremes.* In the ſeries a, ar, ar^2, ar^3, &c. if y be the laſt term, then ſhall the four laſt terms of the ſeries be $y, \frac{y}{r}, \frac{y}{r^2}, \frac{y}{r^3}$; now it is plain that $a \times y = ar \times \frac{y}{r} = ar^2 \times \frac{y}{r^2} = ar^3 \times \frac{y}{r^3}$, &c.

§ 67. "*The ſums of a ſeries of geometrical proportionals wanting the firſt term, is equal to the ſum of all but the laſt term multiplied by the common ratio.*"

For $ar + ar^2 + ar^3, \text{\&c.} + \frac{y}{r^3} + \frac{y}{r^2} + \frac{y}{r} + y$

$$= r \times \overline{a + ar + ar^2, \text{\&c.} + \frac{y}{r^4} + \frac{y}{r^3} + \frac{y}{r^2} + \frac{y}{r}}.$$

Therefore if s be the ſum of the ſeries, $s - a$ will be equal to $\overline{s - y} \times r$; that is $s - a = sr - yr$, or $sr - s = yr - a$, and $s = \frac{yr - a}{r - 1}$*.

§ 68. Since the exponent of r is always increaſing from the ſecond term, if the number of terms be n, in the laſt term its exponent will be $n - 1$. Therefore $y = ar^{n-1}$; and $yr = ar^{n-1+1} = ar^n$; and $s = \left(\frac{yr - a}{r - 1}\right) = \frac{ar^n - a}{r - 1}$. So that having the firſt term of the ſeries, the number of the terms, and the common ratio, you may eaſily find the ſum of all the terms.

If it is a decreaſing ſeries whoſe ſum is to be found, as of $y + \frac{y}{r} + \frac{y}{r^2} + \frac{y}{r^3}$, &c. $+ ar^3 + ar^2 + ar + a$, and the number of the terms be ſuppoſed infinite, then ſhall a, the laſt term, be equal to nothing. For, becauſe n, and conſequently r^{n-1} is infinite, $a = \frac{y}{r^{n-1}} = 0$. The ſum of ſuch a ſeries $s = \frac{yr}{r-1}$; which is a finite ſum, though the number of terms be infinite.

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* See the Rules in the following Chapter.

Thus $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} +$, &c. $= \frac{1 \times 2}{2-1} = 2$.

and $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} +$, &c. $= \frac{1 \times 3}{3-1} = \frac{3}{2}$.

CHAP. X.

Of EQUATIONS that involve only one unknown Quantity.

§ 69. AN *equation* is "*a propoſition aſſerting the equality of two quantities.*" It is expreſſed moſt commonly by ſetting down the quantities, and placing the ſign (=) between them.

An equation gives the value of a quantity, when that quantity is alone on one ſide of the equation: and that value is known, if all thoſe that are on the other ſide are known. Thus if I find that $x = \frac{4 \times 6}{3} = 8$, I have a known value of x. Theſe are the laſt concluſions we are to ſeek in queſtions to be reſolved; and if there be only one unknown quantity in a given equation, and only one dimenſion of it, ſuch a value may always be found by the following Rules.

RULE I.

§ 70. "*A quantity may be tranſpoſed from one ſide of the equation to the other, if you change its ſign.*"

For to take away a quantity from one ſide, and to place it with a contrary ſign on the other ſide, is to ſubtract it from both ſides; and it is certain, that "when from equal quantities you ſubtract the ſame quantity, the remainders muſt be equal."

By this Rule, when the known and unknown quantities are mixed in an equation, you may ſeparate them by bringing all the unknown to one ſide, and the known to the other ſide of the equation; as in the following Examples.

Suppoſe $5x+50=4x+56$,
By tranſpoſit. $5x-4x=56-50$, or, $x=6$.
And if $2x+a=x+b$,
$2x-x=b-a$, or, $x=b-a$.

RULE II.

§ 71. "*Any quantity by which the unknown quantity is multiplied may be taken away, if you divide all the other quantities on both ſides of the equation by it.*"

For that is to divide both ſides of the equation by the ſame quantity; and when you divide equal quantities by the ſame quantity, the quotients muſt be equal. Thus,

If $ax=b$,
then $x=\frac{b}{a}$.

And if $3x+12=27$,
by Rule 1. $3x=27-12=15$,
and by Rule 2. $x=\frac{15}{3}=5$.

Alſo if $ax+2ba=3cc$,
by Rule 1. $ax=3cc-2ba$,
and by Rule 2. $x=\frac{3cc}{a}-2b$.

RULE III.

§ 72. *If the unknown quantitity is divided by any quantity, that quantity may be taken away if you multiply all the other members of the equation by it.*" Thus,

If $\frac{x}{b}=b+5$,
then ſhall $x=bb+5b$.

If $\frac{x}{5}+4=10$,
$x+20=50$,
and by Rule. 1. $x=50-20=30$.

If

$$\text{If } \frac{4x}{3}+24=2x+6,$$
$$4x+72=6x+18,$$
by Rule 1. $72-18=6x-4x$, or $54=2x$.

and by Rule 2. $x=\frac{54}{2}=27$.

By this Rule an equation, whereof any part is a fraction, may be reduced to an equation that ſhall be expreſſed by integers. If there are more fractions than one in the given equation, you may, by reducing them to a common denominator, and then multiplying all the other terms by that denominator, abridge the calculation thus;

$$\text{If } \frac{x}{5}+\frac{x}{3}=x-7,$$
$$\text{then } \frac{3x+5x}{15}=x-7,$$
and by this Rule $3x+5x=15x-105$,

and by Rule 1 and 2. $x=\frac{105}{7}=15$.

RULE IV.

§ 73. "*If the member of the equation that involves the unknown quantity be a* ſurd *root, then the equation is to be reduced to another that ſhall be free from any ſurd, by bringing that member firſt to ſtand alone upon one ſide of the equation, and then taking away the radical ſign from it, and raiſing the other ſide of the equation to the power denominated by the ſurd*".

Thus if $\sqrt{4x+16}=12$,

$4x+16=144$,

and $4x=144-16=128$,

and $x=\frac{128}{4}=32$.

If $\sqrt{ax+b^2}-c=d$,

then $\sqrt{ax+b^2}=d+c$,

$ax+b^2=d^2+2dc+c^2$,

and $x=\frac{d^2+2dc+c^2-b^2}{a}$.

If $\sqrt[3]{a^2x-b^2x}=a$,

then $a^2x-b^2x=a^3$,

and $x=\frac{a^3}{a^2-b^2}$.

RULE V.

§ 74. "*If that ſide of the equation that contains the unknown quantity be a compleat ſquare, cube, or other power; then extract the ſquare root, cube root, or the root of that power, from both ſides of the equation, and thus the equation ſhall be reduced to one of a lower degree*".

If $x^2+6x+9=20$,

then $x+3=\pm\sqrt{20}$,

and $x=\pm\sqrt{20}-3$.

$$\text{If } x^2+ax+\frac{a^2}{4}=b^2,$$
$$\text{then } x+\frac{a}{2}=\pm b,$$
$$\text{and } x=\pm b-\frac{a}{2}.$$

$$\text{If } x^2+14x+49=121,$$
$$\text{then } x+7=\pm 11,$$
$$\text{and } x=\pm 11-7=4, \text{ or } -18.$$

RULE VI.

§ 75. "*A proportion may be converted into an equation, asserting the product of the extreme terms equal to the product of the mean terms; or any one of the extremes equal to the product of the means divided by the other extreme.*"

$$\text{If } 12-x : \frac{x}{2} :: 4 : 1,$$
$$\text{then } 12-x=2x \ldots\ldots 3x=12 \ldots\ldots \text{and } x=4.$$
$$\text{Or if } 20-x : x :: 7 : 3,$$
$$\text{then } 60-3x=7x \ldots\ldots 10x=60 \ldots\ldots \text{and } x=6.$$

RULE VII.

§ 76. "*If any quantity be found on both ſides of the equation with the ſame ſign prefixt, it may be taken away from both:*" "*Alſo, if all the quantities in the equation are multiplied or divided*

divided by the ſame quantity, it may be ſtruck out of them all." Thus,

If $3x+b=a+b \ldots 3x=a \ldots$ and $x=\frac{a}{3}$.

If $3ax+5ab=8ac \ldots 3x+5b=8c \ldots$ and $x=\frac{8c-5b}{3}$.

If $\frac{7x}{3}+\frac{8}{3}=\frac{16}{3} \ldots 2x+8=16 \ldots$ and $x=4$.

RULE VIII.

§ 77. "*Inſtead of any quantity in an equation you may ſubſtitute another equal to it.*"

Thus, if $3x+y=24$,
and $y=9$;
then $3x+9=24 \ldots x=\frac{24-9}{3}=5$.

If $3y+5x=120$,
and $y=5x$;
then $15x+5x(=20x)=120$,
and $x=\frac{120}{20}=6$.

The further improvements of this Rule ſhall be taught in the following chapter.

CHAP. IX.

Of the Solutions of queſtions that produce ſimple equations.

SIMPLE equations are thoſe "wherein the unknown quantity is only of one dimenſion:" In the ſolution of which we are to obſerve the following directions.

DIRECTION I.

§ 78. "*After forming a diſtinct idea of the queſtion propoſed, the unknown quantities are to be expreſſed by letters, and the particulars to be tranſlated from the common language into the algebraic manner of expreſſing them, that is, into ſuch equations as ſhall expreſs the relations or properties that are given of ſuch quantities.*"

Thus, if the ſum of two quantities muſt be 60, that condition is expreſſed thus, $x+y=60$.

If their difference muſt be 24, that condition gives . . . . $x-y=24$.

If their product muſt be 1640, then $xy=1640$.

If their quotient muſt be 6, then . . . $\frac{x}{y}=6$.

If their proportion is as 3 to 2, then $x:y::3:2$, or $2x=3y$; becauſe the product of the extremes is equal to the product of the mean terms.

DIRECTION II.

§ 79. "*After an equation is formed, if you have one unknown quantity only, then, by the Rules of the preceding Chapter, bring it to ſtand alone on one ſide, ſo as to have only known quantities on the other ſide:*" *thus you ſhall diſcover its value.*

EXAMPLE.

A perſon being aſked what was his age, anſwered that $\frac{3}{4}$ *of his age multiplied by* $\frac{1}{12}$ *of his age gives a product equal to his age.* Qu. *what was his age?*

It appears from the queſtion, that if you call his age x, then ſhall . . . $\frac{3x}{4}\times\frac{x}{12}=x$,

that is . . . $\frac{3x^2}{48}=x$;

and by Rule 3. . . . $3x^2=48x$,

and by Rule 7. . . . $3x=48$,

whence by Rule 2. . . . $x=16$.

DIRECTION III.

§ 80. "*If there are two unknown quantities, then there muſt be two equations ariſing from the conditions of the queſtion: Suppoſe the quantities* x *and* y*; find a value of* x *or* y*, from*

each

each of the equations, and then by putting theſe two values equal to each other, there will ariſe a new equation involving one unknown quantity; which muſt be reduced by the Rules of the former Chapter."

EXAMPLE I.

Let the ſum of two quantities be s, and their difference d. Let s and d be given, and let it be required to find the quantities themſelves. Suppoſe them to be x and y, then, by the ſuppoſition,

$$x+y=s$$
$$x-y=d$$

$$\text{whence} \left\{ \begin{array}{l} x=s-y \\ x=d+y \end{array} \right.$$

$$\text{and } d+y=s-y$$
$$2y=s-d$$
$$y=\frac{s-d}{2}$$
$$\text{and } x=\frac{s+d}{2}.$$

EXAMPLE II.

Let it be required to find two numbers whoſe ſum is s, and their proportion as a to b. Let the numbers be x and y, then ſhall

Suppoſ. $\begin{cases} x+y=s \\ x:y::a:b \end{cases}$

$$bx=ay$$

$$x=\frac{ay}{b}$$

$$x=s-y$$

$$\frac{ay}{b}=s-y$$

$$\frac{ay}{b}+y=s$$

$$ay+by=bs$$

$$\overline{a+b}\times y=bs$$

$$y=\frac{bs}{a+b}$$

$$x=\frac{ay}{b}=\frac{as}{a+b}.$$

EXAMPLE III.

A privateer running at the rate of 10 *miles an hour, diſcovers a ſhip* 18 *miles off making way at the rate of* 8 *miles an hour: It is demanded how many miles the ſhip can run before ſhe be overtaken?*

Let the number of miles the ſhip can run before ſhe be overtaken be called x; and the number of miles the privateer muſt run before ſhe come up with the ſhip, be y; then ſhall (by *Supp.*) .... $y=x+18$ .... and $x:y::8:10$, whence $10x=8y$ .... $x=\frac{4y}{5}$.... and $x=y-18$.

Whence

Whence $y-18=\frac{4y}{5}$, and $y=90\ldots x=y-18=72$.

To find the time, ſay, if 8 miles give 1 hour; 72 miles will give 9 hours.—Thus, $8:1::72:9$.

EXAMPLE IV.

Suppoſe the diſtance between London *and* Edinburgh *to be* 360 *miles, and that a courier ſets out from* Edinburgh *running at the rate of* 10 *miles an hour; another ſets out at the ſame time from* London, *and runs* 8 *miles an hour. It is required to know where they will meet?* Suppoſe the courier that ſets out from *Edinburgh* runs x miles, and the other y miles before they meet; then ſhall

$$\text{by ſuppoſ.}\ \begin{cases} x+y=360 \\ x:y::5:4 \end{cases}$$

$$x=\frac{5y}{4}$$

$$x=360-y$$

$$\frac{5y}{4}=360-y$$

$$\frac{5y}{4}+y=360$$

$$9y=1440$$

$$y=\frac{1440}{9}=160$$

$$x=360-y=200.$$

EXAMPLE V.

Two perſons diſcourſing of their revenues, ſays A, *if* B *would yield him a poſt he has of* 25 l. *a year,*

year, their revenues would be equal: Says B, *if* A *would give him a place he holds of* 22l. per annum, *the revenue of* B *would be double that of* A. Qu. *their revenues?*

Let the revenue of *A* be called *x*, that of *B*, *y*; then,

$$\text{by ſupp.}\ \begin{cases} x+25=y-25 \\ y+22=2x-44 \end{cases}$$

$$\begin{aligned} y&=x+25+25=x+50 \\ y&=2x-44-22=2x-66 \\ 2x-66&=x+50 \\ x&=66+50=116 \\ y&=x+50=166. \end{aligned}$$

EXAMPLE VI.

A gentleman diſtributing money among ſome poor people, found he wanted 10s. *to be able to give* 5s. *to each; therefore he gives each* 4s. *only, and finds that he has* 5s. *left.* Qu. *the number of ſhillings and poor people?*

Call the number of the poor *x*, and the number of ſhillings *y*; then,

$$\text{by ſupp.}\ \begin{cases} 5x=y+10 \\ 4x=y-5 \end{cases}$$

$$\begin{aligned} y&=5x-10 \\ y&=4x+5 \\ 5x-10&=4x+5 \\ 5x-4x&=15 \\ x&=15 \\ y&=4x+5=65 \end{aligned}$$

EXAMPLE VII.

Two merhants were copartners; the ſum of their ſtock was 300 l. *One of their ſtocks continued in company* 11 *months; but the other drew out his ſtock in* 9 *months; when they made up their accounts they divided the gain equally.* Qu. *What was each man's ſtock?* Suppoſe the ſtock of the firſt to be x, and the ſtock of the other to be y; then,

by ſupp. $\begin{cases} x+y=300 \\ 11x=9y \end{cases}$

$$x=\frac{9y}{11}=300-y$$
$$11y+9y=3300$$
$$20y=3300$$
$$y=\frac{3300}{20}=165 .. x=300-y=135.$$

EXAMPLE. VIII.

There are two numbers whoſe ſum is the 6th part of their product, and the greater is to the leſſer as 3 *to* 2. Qu. *What are theſe numbers?* Call them x and y; then,

ſupp. $\begin{cases} x+y=\frac{xy}{6} \\ x:y::3:2 \end{cases}$

$$yx=6x+6y$$
$$yx-6x=6y$$
$$\overline{y-6}\times x=6y$$
$$x=\frac{6y}{y-6}$$
$$x=\frac{3y}{2}, \text{ whence}*$$

$$*\frac{6y}{y-6}=\frac{3y}{2}$$
$$12y=3yy-18y$$
$$30y=3yy$$
$$30=3y$$
$$y=\frac{30}{3}=10$$
$$x=\frac{3\times 10}{2}=15.$$

DIRECTION IV.

§ 81. "*When in one of the given equations, the unknown quantity is of one dimenſion, and in the other of a higher dimenſion; you muſt find a value of the unknown quantity from that equation where it is of one dimenſion, and then raiſe that value to the power of the unknown quantity in the other equation; and by comparing it, ſo involved, with the value you deduce from that other equation, you ſhall obtain an equation that will have only one unknown quantity, and its powers.*"

That is, when you have two equations of different dimenſions, if you cannot reduce the higher to the ſame dimenſion with the lower, you muſt raiſe the lower to the ſame dimenſion with the higher.

EXAMPLE IX.

The ſum of two quantities, and the difference of their ſquares, being given, to find the quantities.

Suppoſe them to be x and y, their ſum s, and difference of their ſquares d. Then,

$$\begin{cases} x+y=s \\ x^2-y^2=d \end{cases}$$

$$x=s-y$$

$$x^2=s^2-2sy+y^2$$

$$x^2=d+y^2$$

$$d+y^2=s^2-2sy+y^2$$

$d=s^2-2sy$, whence *

$$* \; 2sy=s^2-d$$

$$y=\frac{s^2-d}{2s}$$

and $x=\dfrac{s^2+d}{2s}$.

EXAMPLE

EXAMPLE X.

Let the proportion of two numbers and the ſum of their ſquares be given, and let it be required to find the numbers themſelves. Suppoſe their proportion to be the ſame as that of a to b, and let the ſum of their ſquares be c; that is, let

$$\begin{cases} x : y :: a : b \\ x^2 + y^2 = c \end{cases}$$

then $x = \frac{ay}{b}$,

and $x^2 = \frac{a^2y^2}{b^2}$;

but $x^2 = c - y^2$,

whence $c - y^2 = \frac{a^2y^2}{b^2}$

$$b^2y^2 + a^2y^2 = cb^2$$

$$\overline{a^2 + b^2} \times y^2 = cb^2$$

$$y^2 = \frac{cb^2}{a^2 + b^2}$$

$y = \sqrt{\frac{cb^2}{a^2 + b^2}}$, and $x = \sqrt{\frac{ca^2}{a^2 + b^2}}$.

EXAMPLE XI.

Let the proportion of two numbers be that of a to b, and the difference of their cubes be d. Qu. *What are the numbers?* Then,

$$\begin{cases} x:y::a:b \\ x^3-y^3=d \end{cases}$$

$$x=\frac{ay}{b}, \text{ and } x^3=\frac{a^3y^3}{b^3};$$

$$\text{but } x^3=d+y^3$$

$$\text{whence } \quad d+y^3=\frac{a^3y^3}{b^3},$$

$$\text{and } a^3y^3-b^3y^3=db^3$$

$$y^3=\frac{db^3}{a^3-b^3}$$

$$y=\sqrt[3]{\frac{db^3}{a^3-b^3}}$$

$$\text{and } x=\sqrt[3]{\frac{da^3}{a^3-b^3}}.$$

DIRECTION V.

§ 82. "*If there are three unknown quantities, there muſt be three equations in order to determine them, by comparing which you may, in all caſes, find two equations involving only two unknown quantities; and then, by* Direction 3, *from theſe two you may deduce an equation involving only one unknown quantity; which may be reſolved by the Rules of the laſt Chapter.*"

From three equations involving any three unknown quantities, x, y, and z, to deduce two equations involving only two unknown quantities, the following Rule will always ſerve.

RULE.

"*Find three values of x from the three given equations; then, by comparing the firſt and ſecond value, you will find an equation involving only y and z; again, by comparing the firſt and third, you will find another equation involving only y and z;" and laſtly, thoſe equations are to be reſolved by* Direction 3.

EXAMPLE XII.

Suppoſe

$$\left.\begin{array}{l} x+y+z=12 \\ x+2y+3z=20 \\ \frac{x}{3}+\frac{y}{2}+z=6 \end{array}\right\} \begin{array}{c}\text{then,}\\ x=\end{array}\left\{\begin{array}{l} 12-y-z \\ 20-2y-3z \\ 18-\frac{3y}{2}-3z \end{array}\right\} \begin{array}{l}\text{1ſt}\\ \text{2d}\\ \text{3d}\end{array}\left\{\text{value.}\right.$$

$$12-y-z=20-2y-3z$$

$$12-y-z=18-\frac{3y}{2}-3z$$

Theſe two laſt equations involve only y and z, and are to be reſolved, by *Direction* 3, as follows.

$2y-$

$$\left\{\begin{array}{l} 2y- y+3z-z=20-12=8 \\ y+2z=8 \end{array}\right.$$

$$36-3y-6z=24-2y-2z$$

$$12= y+4z$$

$$\text{whence}\quad y=\left\{\begin{array}{l} 8-2z\ ..\ \text{1ſt value.} \\ 12-4z\ ..\ \text{2d value.} \end{array}\right.$$

$$8-2z=12-4z$$

$$2z=12-8=4$$

$$\text{and}\quad z=2$$

$$y(=8-2z)=4$$

$$x(=12-y-z=6.$$

§ 83. This method is general, and will extend to all equations that involve three unknown quantities: but there are often eaſier and ſhorter methods to deduce an equation involving one unknown quantity only; which will be beſt learned by practice.

EXAMPLE XIII.

$$\text{Suppoſing}\left\{\begin{array}{l} x+y+z=26 \\ x-y=4 \\ x-z=6 \end{array}\right.$$

$$\text{by addition}\quad 3x=36$$

$$\left\{\begin{array}{l} x=\frac{36}{3}=12 \\ y=x-4=8 \\ z=x-6=6. \end{array}\right.$$

EX-

EXAMPLE XIV.

$$\text{Suppoſing}\begin{cases} x+y=a \\ x+z=b \\ y+z=c \end{cases}$$

$$\begin{aligned} x&=a-y \\ a-y+z&=b \\ y+z&=c \\ \hline a+0+2z&=b+c \\ 2z&=b+c-a \end{aligned}$$

$$\begin{cases} z=\dfrac{b+c-a}{2} \\ y(=c-z)=\dfrac{c+a-b}{2} \\ x(=a-y)=\dfrac{a+b-c}{2}. \end{cases}$$

§ 84. It is obvious from the 3d and 5th Directions, in what manner you are to work if there are four, or more, unknown quantities, and four, or more, equations given. By comparing the given equations, you may always at length diſcover an equation involving only one unknown quantity; which, if it is a ſimple equation, may always be reſolved by the Rules of the laſt Chapter. We may conclude then, that "When there are as many ſimple equations given as quantities required, theſe quantities may be diſcovered by the application of the preceding Rules."

§ 85. "If indeed there are more quantities required than equations given, then the queſtion is not limited to determinate quantities; but is capable of an infinite nnmber of ſolutions." And, "If there are more equations given than there are quantities required, it may be impoſſible to find the quantities that will anſwer the conditions of the queſtion;" becauſe ſome of theſe conditions may be inconſiſtent with others.

CHAP. XII.

Containing ſome General Theorems for the exterminating unknown Quantities in given Equations.

IN the following *Theorems*, we call thoſe coefficients of the "*ſame order*" that are prefixt to the ſame unknown quantities in the different equations. Thus, in *Theor.* 2. a, d, g, are of the ſame order, being the coefficients of x: alſo b, e, h, are of the ſame order, being the coefficients of y: and thoſe are of the ſame order that affect no unknown quantity.

But thoſe are called "*oppoſite*" coefficients that are taken each from a different equation, and from a different order of coefficients: As, a, e, and d, b, in the firſt Theorem; and a, e, k, in the ſecond; alſo, a, b, f; and d, b, k, &c.

THEOREM I.

§ 86. Suppoſe that two equations are given, involving two unknown quantities, as

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

then ſhall $y = \frac{af - dc}{ae - db}$.

Where the numerator is the difference of the products of the oppoſite coefficients in the orders in which y is not found, and the denominator is the difference of the products of the oppoſite coefficients taken from the orders that involve the two unknown quantities.

For, from the firſt equation, it is plain that

$$ax = c - by \,..\, \text{and } x = \frac{c - by}{a},$$

from the 2d, $dx = f - ey$ .. and $x = \frac{f - ey}{d}$,

therefore $\frac{c - by}{a} = \frac{f - ey}{d}$, and $cd - dby = af - aey$;

whence $aey - dby = af - cd$,

and $y = \frac{af - cd}{ae - db}$;

after the ſame manner, $x = \frac{ce - bf}{ae - db}$.

EX-

EXAMPLE I.

$$\text{Supp.}\begin{cases}5x+7y=100\\3x+8y=\ 80\end{cases}$$

$$\text{then } y=\frac{5\times80-3\times100}{5\times8-3\times7}=\frac{100}{19}=5\tfrac{5}{19};$$

$$\text{and } x=\frac{240}{19}=12\tfrac{12}{19}.$$

EXAMPLE II.

$$\begin{cases}4x+8y=\ 90\\3x-2y=160\end{cases}$$

$$y=\frac{4\times160-3\times90}{4\times-2-3\times8}=\frac{640-270}{-8-24}=\frac{370}{-32}=-11\tfrac{9}{16}.$$

THEOREM II.

§ 87. Suppoſe now that there are three unknown quantities and three equations, then call the unknown quantities x, y, and z.

$$\text{Thus}\begin{cases}ax+by+cz=m\\dx+ey+fz=n\\gx+hy+kz=p\end{cases}$$

$$\text{Then ſhall } z=\frac{aep-ahn+dhm-dbp+gbn-gem}{aek-ahf+dhc-dbk+gbf-gec}.$$

Where the numerator conſiſts of all the different products that can be made of three oppoſite coefficients taken from the orders in which z is not found; and the denominator conſiſts of all the products that can be made of the three oppoſite coefficients taken from the orders that involve the three unknown quantities. For, from the laſt it appears, that

$$y = \frac{an - afz - dm + dcz}{ae - db}, \text{ and that}$$

$$y = \frac{ap - akz - gm + gcz}{ah - gb}; \text{ therefore}$$

$$\frac{an - afz - dm + dcz}{ae - db} = \frac{ap - akz - gm + gcz}{ah - gb}, \text{ and}$$

$$\overline{an - afz - dm + dcz} \times ah - gb \times \overline{an - afz} + gbdm - gbdcz = \overline{ap - gm - akz + gcz} \times ae - db \times \overline{ap - akz} + gbdm - gbdcz.$$

Take $gbdm - gbdcz$ from both ſides, and divide by a, ſo ſhall

$$\overline{an - dm - afz + dcz} \times h - gbn + gbfz = \overline{ap - gm - akz + gcz} \times e - dbp + dbkz.$$

Tranſpoſe and divide, ſo ſhall you find $z = \frac{aep - ahn + dhm - dbp + gen - gem}{aek - ahf + dhc - dbk + gbf - gec}$. The values of x and y are found after the ſame manner, and have the ſame denominator. *Ex gr.* $y = \frac{afp - akn + dkm - dep + gen - gfm}{ack - ahf + dhc - dbk + gbf - gec}$.

If any term is wanting in any of the three given equations, the values of z and y will be found more ſimple. Suppoſe, for example, that f and k are equal to nothing, then the term fz will vaniſh in the ſecond equation, and kz in the third, and $z = \frac{aep - anh + dhm - dbp + gnb - gem}{dhc - gec}$;

$$y = \frac{gcn - dep}{dhc - gec}.$$

If four equations are given, involving four unknown quantities, their values may be found much after the ſame manner, by taking all the products that can be made of four oppoſite coefficients, and always prefixing contrary ſigns to thoſe that involve the products of two oppoſite coefficients.

CHAP. XIII.

Of Quadratic EQUATIONS.

§ 88. IN the ſolution of any queſtion where you have got an equation that involves one unknown quantity, but involves at the ſame time the ſquare of that quantity, and the product of it multiplied by ſome known quantity, then you have what is called a *Quadratic equation*; which may be reſolved by the following

RULE.

1. "*Tranſport all the terms that involve the unknown quantity to one ſide, and the known terms to the other ſide of the equation.*

2. *If the ſquare of the unknown quantity is multiplied by any coefficient, you are to divide all the terms by that coefficient, that the coefficient*

of the ſquare of the unknown quantity may be unit.

3. *Add to both ſides the ſquare of half the coefficient prefixed to the unknown quantity itſelf, and the ſide of the equation that involves the unknown quantity will then be a compleat ſquare.*

4. *Extract the ſquare root from both ſides of the equation; which you will find, on one ſide, always to be the unknown quantity with half the foreſaid coefficient ſubjoined to it; ſo that by tranſpoſing this half you may obtain the value of the unknown quantity expreſſed in known terms.*" Thus,

Suppoſe $y^2 + ay = b$,

Add the ſquare of $\frac{a}{2}$ to both ſides $\Big\}\; y^2 + ay + \frac{a^2}{4} = b + \frac{a^2}{4}$,

Extract the root, $y + \frac{a}{2} = \pm\sqrt{b + \frac{a^2}{4}}$,

Tranſpoſe $\frac{a}{2}$, $y = \pm\sqrt{b + \frac{a^2}{4}} - \frac{a}{2}$.

§ 89. The ſquare root of any quantity, as $+aa$, may be $+a$, or $-a$; and hence, "All quadratic equations admit of two ſolutions." In the laſt example, after finding that $y^2 + ay + \frac{a^2}{4} = b + \frac{a^2}{4}$, it may be inferred that $y + \frac{a}{2} = +\sqrt{b + \frac{a^2}{4}}$, or to $-\sqrt{b + \frac{a^2}{4}}$; ſince

$-\sqrt{b+\frac{a^2}{4}}\times-\sqrt{b+\frac{a^2}{4}}$ gives $b+\frac{a^2}{4}$, as well as $+\sqrt{b+\frac{a^2}{4}}\times+\sqrt{b+\frac{a^2}{4}}$. There are therefore two values of y; the one gives $y=+\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$, the other $y=-\sqrt{b+\frac{a^2}{4}}-\frac{a}{2}$.

§ 90. Since the ſquares of all quantities are poſitive, it is plain that "The ſquare root of a negative quantity is imaginary, and cannot be aſſigned." Therefore there are ſome quadratic equations that cannot have any ſolution. For example,

Suppoſe $y^2-ay+3a^2=0$,
then $y^2-ay=-3a^2$;
add $\frac{a^2}{4}$ to both, $y^2-ay+\frac{a^2}{4}=-3a^2+\frac{a^2}{4}=-\frac{11a^2}{4}$,
extract the root, $y-\frac{a}{2}=\pm\sqrt{-\frac{11a^2}{4}}$,
and $y=\frac{a}{2}\pm\sqrt{-\frac{11a^2}{4}}$:

whence the two values of y muſt be imaginary or impoſſible, becauſe the root of $-\frac{11a^2}{4}$ cannot poſſibly be aſſigned.

But of this we ſhall treat more fully in the Second Part.

Suppoſe that the quadratic equation propoſed to be reſolved is $y^2 - ay = b$;

$$\text{then } y^2 - ay + \frac{a^2}{4} = b + \frac{a^2}{4},$$

$$y - \frac{a}{2} = \pm\sqrt{b + \frac{a^2}{4}},$$

$$\text{and } y = \frac{a}{2} \pm \sqrt{b + \frac{a^2}{4}}.$$

If the ſquare root of $b + \frac{a^2}{4}$ cannot be extracted exactly, you muſt, in order to determine the value of y, nearly approximate to the value of $\sqrt{b + \frac{a^2}{4}}$, by the Rules in *Chap.* 8. The following examples will illuſtrate the Rule for quadratic equations.

EXAMPLE I.

To find that number, which if you multiply by 8, *the product ſhall be equal to the ſquare of the ſame number, having* 12 *added to it.*

Call the number y; then

$$y^2 + 12 = 8y,$$

$$\text{tranſp. } y^2 - 8y = -12,$$

$$\text{Add the ſq. of } 4,\ y^2 - 8y + 16 = -12 + 16 = 4,$$

$$\text{extract the root } y - 4 = \pm 2,$$

$$\text{tranſp. } y = 4 \pm 2 = 6, \text{ or } 2.$$

EXAMPLE II.

To find a number ſuch that if you ſubtract it from 10, *and multiply the remainder by the nnmber itſelf, the product ſhall give* 21.

Call it y; then

$$\overline{10-y}\times y=21.$$

that is, $10y-yy=21$;

tranſp. $y^2-10y=-21$,

add the ſq. of $5, y^2-10y+25=-21+25=4$,

extr. the ſq. root $y-5=\pm\sqrt{4}=\pm 2$,

and $y=5\pm 2=7$, or 3.

EXAMPLE III.

The ſum of two quantities is a, *their product* b. Qu. *What are the quantities?*

$$\text{Suppoſe}\begin{cases} x+y=a\ldots\text{then } x=a-y, \\ xy=b\ldots\ldots\text{then } x=\dfrac{b}{y}, \end{cases}$$

therefore $a-y=\dfrac{b}{y}$,

and $ay-y^2=b$;

tranſp. $y^2-ay=-b$,

add $\dfrac{a^2}{4}\ldots y^2-ay+\dfrac{a^2}{4}=-b+\dfrac{a^2}{4}$,

extract $\surd$, $y-\dfrac{a}{2}=\pm\sqrt{-b+\dfrac{a^2}{4}}$,

and $y=\dfrac{a}{2}\pm\sqrt{-b+\dfrac{a^2}{4}}$,

$x(=a-y)=\dfrac{a}{2}\mp\sqrt{-b+\dfrac{a^2}{4}}$.

EXAMPLE IV.

The ſum of two quantities is a, and the ſum of their ſquares b. Qu. *the quantities?*

Suppoſe $\left\{\begin{array}{l} x+y=a \ldots \text{then } x=a-y, \\ x^2+y^2=b \ldots\ldots\ldots x^2=b-y^2, \end{array}\right.$

involv. $x^2=a^2-2ay+y^2$,

whence $a^2-2ay+y^2=b-y^2$;

$\left.\begin{array}{l}\text{tranſp.}\\ \text{and di-}\\ \text{vide}\end{array}\right\{ \begin{array}{l} 2y^2-2ay=b-a^2, \\ y^2-\ ay=\dfrac{b-a^2}{2}, \end{array}$

add $\dfrac{a^2}{4}$, $y^2-ay+\dfrac{a^2}{4}=\dfrac{b-a^2}{2}+\dfrac{a^2}{4}=\dfrac{2b-a^2}{4}$,

extr. $\surd$, $y-\dfrac{a}{2}=\pm\sqrt{\dfrac{2b-a^2}{4}}$; and $y=\dfrac{a}{2}\pm\sqrt{\dfrac{2b-a^2}{4}}$; $x\ (=a-y)=\dfrac{a}{2}\mp\sqrt{\dfrac{2b-a^2}{4}}$.

Or thus, $y=\dfrac{a\pm\sqrt{2b-a^2}}{2}$, and $x=\dfrac{a\pm\sqrt{2b-a^2}}{2}$.

EXAMPLE V.

A company dining together in an inn, find their bill amounts to 175 *ſhillings; two of them were not allowed to pay, and the reſt found that their ſhares amounted to* 10 s. *a man more than if all had paid.* Qu. *How many were in company?*

Suppoſe their number x; then if all had paid each man's ſhare would have been $\frac{175}{x}$, ſeeing $x-2$ is the number of thoſe that pay. It is therefore, by the queſtion,

$$\frac{175}{x-2}-\frac{175}{x}=10,$$

and $175x-175x+350=10x^2-20x$;

that is, $10x^2-20x=350$,

and $x^2-2x=35$;

add 1 .. $x-2x+1=35+1=36$,

extr. $\sqrt{\ }$ .., $x-1=\pm 6$,

$x=1\pm 6=7$, or -5.

It is obvious that the poſitive value 7 gives the ſolution of the queſtion; the negative value -5 being, in the preſent caſe, uſeleſs.

EXAMPLE VI.

There are three numbers in continual geometrical proportion; the ſum of the firſt and ſecond is 10, *and the difference of the ſecond and* 3d *is* 24. Qu. *the numbers?*

Let the firſt be x, and the ſecond will be $10-x$, and the third, $34-x$; therefore,

$$x : 10-x :: 10-x : 34-x,$$

and $34x-x^2=100-20x+x^2$;

tranſp. and divid. $\left\{\begin{array}{l}54x=100+2x^2,\\ x^2-27x=-50,\end{array}\right.$

add $\frac{27}{2}\times\frac{27}{2}$ ... $x^2-27x+\frac{729}{4}=\frac{729}{4}-50=\frac{529}{4}$,

extract $\sqrt{\ }$ ... $x-\frac{27}{2}=\pm\sqrt{\frac{529}{4}}=\mp\frac{23}{2}$,

and $x=\frac{27+23}{2}$, or $=\frac{27-23}{2}=25$, or 2.

So the three continued proportionals are

$$2 : 8 : 32, \text{ or}$$
$$25 : -15 : 9.$$

§ 91. Any equation of this form $y^{2m}+ay^m=b$, where the greateſt index of the unknown quantity y is double to the index of y in the other term, may be reduced to a quadratic $z^2+az=b$, by putting $y^m=z$, and conſequently $y^{2m}=z^2$. And this quadratic reſolved as above, gives $z=-\frac{a}{2}\pm\sqrt{b+\frac{a^2}{4}}$.

And ſeeing $y^m=z=-\frac{a}{2}\pm\sqrt{b+\frac{a^2}{4}}$,

$$y=\sqrt[m]{-\frac{a}{2}\pm\sqrt{b+\frac{a^2}{4}}}.$$

EXAMPLE I.

The product of two quantities is a, and the ſum of their ſquares b. Qu. *the quantities?*

$$\text{Supp.}\begin{cases} xy=a \ldots\ldots \text{or, } x=\frac{a}{y},\ x^2=\frac{a^2}{y^2}, \\ x^2+y^2=b \ldots x^2=b-y^2, \end{cases}$$

whence $b-y^2=\frac{a^2}{y^2}$;

mult. by y^2 .. $by^2-y^4=a^2$,

tranſp. $y^4-by^2=-a^2$.

Put now $y^2=z$ ... and conſequently $y^4=z^2$, and it is

z^2-

$$z^2-bz=-a^2;$$

add $\frac{b^2}{4}$, $z^2-bz+\frac{b^2}{4}=\frac{b^2}{4}-a^2$,

ext. $\sqrt{}$, $z-\frac{b}{2}=\pm\sqrt{\frac{b^2}{4}-a^2}$,

and $z=\frac{b}{2}\pm\sqrt{\frac{b^2}{4}-a^2}$; and, ſeeing $y=\sqrt{z}$,

$$y=\pm\sqrt{\frac{b}{2}\pm\sqrt{\frac{b^2}{4}-a^2}}.$$

EXAMPLE II.

To find a number from the cube of which if you ſubtract 19, *and multiply the remainder by that cube, the product ſhall be* 216.

Call the number required x; and then, by the queſtion,

$$\overline{x^3-19}\times x^3=216,$$

$$x^6-19x^3=216.$$

Put $x^3=z \ldots\ldots x^6=z^2$, and it will be

$$z^2-19z+\frac{361}{4}=216+\frac{361}{4}=\frac{1225}{4},$$

and $\sqrt{}$ . . $z-\frac{19}{2}=\pm\frac{35}{2}$;

whence $z=\frac{19\pm35}{2}=27$, or $=-8$.

But $x=\sqrt[3]{z}$; wherefore $x=+3$, or -2.

EX-

EXAMPLE III.

To find the value of x, ſuppoſing that $x^3 - 7x^{\frac{3}{2}} = 8$.

$$\text{Put } x^{\frac{3}{2}} = z, \text{ and } x^3 = z^2;$$
$$\text{then } z^2 - 7z = 8,$$
$$z^2 - 7z + \frac{49}{4} = \frac{81}{4},$$
$$z - \frac{7}{2} = \pm\frac{9}{2},$$
$$z = 8.$$
$$\text{But } x^3 = z^2, \text{ and } x = \sqrt[3]{z^2} = \sqrt[3]{64} = 4.$$

CHAP. XIV.

Of SURDS.

§ 92. IF a leſſer quantity meaſures a greater ſo as to leave no remainder, as $2a$ meaſures $10a$, being found in it five times, it is ſaid to be an *aliquot* part of it, and the greater is ſaid to be a *multiple* of the leſſer. The leſſer quantity in this caſe is the *greateſt common meaſure* of the two quantities; for as it meaſures the greater, ſo it alſo meaſures itſelf, and no quantity can meaſure it that is greater than itſelf.

When a third quantity meaſures any two propoſed quantities, as $2a$ meaſures $6a$ and $10a$, it is ſaid to be a *common meaſure* of theſe quantities; and if no greater quantity meaſure them both, is is called their *greateſt common* meaſure.

Thoſe quantities are ſaid to be *commenſurable* which have any common meaſure; but if there can be no quantity found that meaſures them both, they are ſaid to be *incommenſurable*; and if any one quantity be called *rational*, all others that have any common meaſure with it, are alſo called rational: But thoſe that have no common meaſure with it, are called *irrational* quantities.

§ 93. If any two quantities a and b have any common meaſure x, this quantity x ſhall alſo meaſure their ſum or difference $a \mp b$. Let x be found in a as many times as unit is found in m, ſo that $a = mx$; and in b, as many times as unit is found in n, ſo that $b = nx$; then ſhall $a \mp b = mx \mp nx = \overline{m \mp n} \times x$; ſo that x ſhall be found in $a \mp b$, as often as unit is found in $m \mp n$: Now ſince m and n are integer numbers, $m \mp n$ muſt be an integer number or unit, and therefore x muſt meaſure $a \mp b$.

§ 94. It is alſo evident, that if x meaſure any number as a, it muſt meaſure any multiple of that number. If it be found in a as many times as unit is found in m, ſo that $a = mx$, then it will be found in any multiple of a, as na, as many times as unit is found in mn; for $na = mnx$.

§ 95. If two quantities a and b are propoſed, and b meaſure a by the units that are in m (that is, be found in a as many times as unit it found in m) and there be a remainder c; and if x be ſuppoſed to be a common meaſure of a and b, it ſhall be alſo a meaſure of c. For by the ſuppoſition $a = mb + c$, ſince it contains b as many times as there are units in m, and there is c beſides of remainder; therefore $a - mb = c$. Now x is ſuppoſed to meaſure a and b, and therefore it meaſures mb (*Art.* 94.) and conſequently $a - mb$ (*Art.* 93.) which is equal to c.

If c meaſures b by the units in n, and there be a remainder d, ſo that $b = nc + d$, and $b - nc = d$, then ſhall x alſo meaſure d; becauſe it is ſuppoſed to meaſure b, and it has been proved that it meaſures c, and conſequently nc, and $b - nc$ (by *Art.* 94.) which is equal to d. Whence, as after ſubtracting b as often as poſſible from a, the remainder c is meaſured by x; and after ſubtracting c as often as poſſible from b, the remainder d is alſo meaſured by x; ſo, for the ſame reaſon, if you ſubtract d as often as poſſible from c, the remainder (if there be any) muſt ſtill be meaſured by x: and if you proceed, ſtill ſubtracting every remainder from the preceding remainder, till you find ſome remainder which ſubtracted from the preceding leaves no further remainder, but exactly meaſures it, this laſt remainder will ſtill be meaſured by x, any common meaſure of a and b.

§ 96. The laſt of theſe remainders, *viz* that which exactly meaſures the preceding remainder, muſt be a common meaſure of a and b: ſuppoſe that d was this laſt remainder, and that it meaſured c by the units in r, then ſhall $c=rd$, and we ſhall have theſe equations.

$$a = mb + c,$$
$$b = nc + d,$$
$$c = rd.$$

Now it is plain that ſince d meaſures c, it muſt alſo meaſure nc, and therefore muſt meaſure $nc+d$, or b. And ſince it meaſures b and c, it muſt meaſure $mb+c$, or a; ſo that it muſt be a common meaſure of a and b. But further, it muſt be their *greateſt* common meaſure; for every common meaſure of a and b muſt meaſure d, by the laſt article; and the greateſt number that meaſures d, is itſelf, which therefore is the greateſt common meaſnre of a and b.

§ 97. But if, by continually ſubtracting every remainder from the preceding remainder, you can never find one that meaſures that which precedes it, exactly, no quantity can be found that will meaſure both a and b; and therefore they will be *incommenſurable* to each other.

For if there was any common meaſ[illegible]e quantities, as x, it would neceſſarily meaſure all the remainders c, d, &c. For it would meaſure $a - mb$, or c, and conſequently $b - nc$; or d; and ſo on. Now theſe remainders decreaſe in ſuch a manner, that they will neceſſarily become at length leſs than x, or any aſſignable quantity: for c muſt be leſs than $\frac{1}{2}a$; becauſe c is leſs than b, and therefore leſs than mb, and conſequently leſs than $\frac{1}{2}c + \frac{1}{2}mb$, or $\frac{1}{2}a$. In like manner d muſt be leſs than $\frac{1}{2}b$; for d is leſs than c, and conſequently leſs than $\frac{1}{2}d + \frac{1}{2}nc$, or $\frac{1}{2}b$. The third remainder, in the ſame manner, muſt be leſs than $\frac{1}{2}c$, which is itſelf leſs than $\frac{1}{2}a$: thus theſe remainders decreaſe ſo, that every one is leſs than the half of that which preceded it next but one. Now if from any quantity you take away more than its half, and from the remainder more than its half, and proceed in this manner, you will come at a remainder leſs than any aſſignable quantity. It appears therefore that if the remainders c, d, &c. never end, they will become leſs than any aſſignable quantity, as x, which therefore cannot poſſibly meaſure them, and therefore cannot be a common meaſure of a and b.

§ 98. In the ſame way, the greateſt common meaſure of two numbers is diſcovered. Unit is a common meaſure of all integer numbers, and two numbers are ſaid to be *prime* to each other, when they have no greater common meaſure than unit; ſuch as 9 and 25. Such always are the leaſt numbers that can be aſſumed in any given proportion; for if theſe had any common meaſure, then the quotients that would ariſe by dividing them by that common meaſure would be in the ſame proportion, and being leſs then the numbers themſelves; theſe numbers would not be the leaſt in the ſame proportion; againſt the ſuppoſition.

§ 99. The leaſt numbers in any proportion always meaſure any other numbers that are in the ſame proportion. Suppoſe a and b to be the leaſt of all integer numbers in the ſame proportion, and that c and d are other numbers in that proportion, then will a meaſure c, and b meaſure d.

For if a and b are not aliquot parts of c and d, then they muſt contain the ſame number of the ſame kind of parts of c and d, and therefore dividing a into parts of c; and b into an equal number of like parts of d; and calling one of the firſt m, and one of the latter n; then as m is to n, ſo will the ſum of all the m's be to the ſum of all the n's; that is, $m : n :: a : b$; therefore a and b will not be the leaſt in the ſame proportion; againſt the ſuppoſition. Therefore a and b muſt be aliquot parts of c and d. Hence we ſee that numbers which are prime to each other are the leaſt in the ſame proportion; for if there were others in the ſame proportion leſs than them, theſe would meaſure them by the ſame number, which therefore would be their common meaſure againſt the ſuppoſition, for we ſuppoſed them to be prime to each other.

§ 100. If two numbers *a* and *b* are prime to one another, and a third number *c* meaſures one of them *a*, it will be prime to the other *b*. For if *c* and *b* were not prime to each other, they would have a common meaſure, which becauſe it would meaſure *c*, would alſo meaſure *a*, which is meaſured by *c*, therefore *a* and *b* would have a common meaſure, againſt the ſuppoſition.

§ 101. If two numbers *a* and *b* are prime to *c*, then ſhall their product *ab* be alſo prime to *c*: For if you ſuppoſe them to have any common meaſure as *d*, and ſuppoſe that *d* meaſures *ab* by the units in *e*, ſo that $de = ab$, then ſhall $d : a :: b : e$. But ſince *d* meaſures *c*, and *c* is ſuppoſed to be prime to *a*, it follows (by *Art.* 100.) that *d* and *a* are prime to each other; and therefore (by *Art.* 99.) *d* muſt meaſure *b*; and yet ſince *d* is ſuppoſed to meaſure *c*, which is prime to *b*, it follows that *d* is alſo prime to *b*: that is, *d* is prime to a number which it meaſures, which is abſurd.

§ 102. It follows from the laſt article, that if *a* and *c* are prime to each other, then a^2 will be prime to *c*: For by ſuppoſing that *a* is equal to *b*, then *ab* will be equal to a^2; and conſequently a^2 will be prime to *c*. In the ſame manner c^2 will be prime to *a*.

§ 103. If two numbers a and b, are both prime to other two c, d, then ſhall the product ab be prime to the product cd; for (by *Art.* 101.) ab will be prime to c and alſo to d, and therefore, by the ſame article, cd will be prime to ab.

§ 104. From this it follows, that if a and c are prime to each other, then ſhall a^2 be prime to c^2, by ſuppoſing, in the laſt, that $a=b$, and $c=d$. It is alſo evident that a^3 will be prime to c^3, and in general any power of a to any power of c whatſoever.

§ 105. Any two numbers, a and b, being given, to find the leaſt numbers that are in the ſame proportion with them, *divide them by their greateſt common meaſure* x, and *the quotients* c *and* d *ſhall be the leaſt numbers in the ſame proportion with* a *and* b.

For if there could be any other numbers in that proportion leſs than c and d, ſuppoſe them to be e and f, and theſe being in the ſame proportion as a and b would meaſure them: And the number by which they would meaſure them, would be greater than x, becauſe e and f are ſuppoſed leſs than c and d, ſo that x would not be the greateſt common meaſure of a and b; againſt the ſuppoſition.

§ 106. Let it be required to find the leaſt number that any two given numbers as a and b can meaſure. Firſt, *if they are prime to each* *other, then their product* ab *is the leaſt number which they can both meaſure.*

For if they could meaſure a leſs number than ab as c, ſuppoſe that c is equal to ma, and to nb; and ſince c is leſs than ab, therefore ma will be leſs than ab, and m leſs than b; and nb being leſs than ab, it follows that n muſt be leſs than a; but ſince $ma = nb$, and conſequently $a : b :: n : m$, and a and b are prime to each other, it would follow that a would meaſure n, and b meaſure m; that is, a greater number would meaſure a leſs, which is abſurd.

But if the numbers a and b are not prime to each other, and their greateſt common meaſure is x, which meaſures a by the units in m, and meaſures b by the units in n, ſo that $a = mx$, and $b = nx$; then ſhall an (which is equal to bm, becauſe $a : b :: mx : nx : m : n$, and therefore $an = bm$) be the leaſt number that a and b can both meaſure. For if they could meaſure any number c leſs than na, ſo that $c = la = kb$, then $a : b :: m : n :: k : l$; and becauſe x is ſuppoſed to be the greateſt common meaſure of a and b, it follows that m and n are the leaſt of all numbers in the ſame proportion, and therefore m meaſures k, and n meaſures l. But as c is ſuppoſed to be leſs than na, that is, la leſs than na, therefore l is leſs than n, ſo that a greater would meaſure a leſſer, which is abſurd. Therefore a and b cannot meaſure any number leſs than na;

na; which they both meaſure, becauſe $na = mb$.

It follows from this reaſoning, that if a and b meaſure any quantity c, the leaſt quantity na, which is meaſured by a and b, will alſo meaſure c. For if you ſuppoſe as before that $c = la$, you will find that n muſt meaſure l, and na muſt meaſure la or c.

§ 107. Let a expreſs any integer number, and $\frac{m}{n}$ any fraction reduced to its loweſt terms, ſo that m and n may be prime to each other, and conſequently $an+m$ alſo prime to n, it will follow that $\overline{an+m}^2$ will be prime to n^2, and conſequently $\frac{\overline{an+m}^2}{n^2}$ will be a fraction in its leaſt terms, and can never be equal to an integer number. Therefore the ſquare of the mixt number $a+\frac{m}{n}$ is ſtill a mixt number, and never an integer. In the ſame manner the cube, biquadrate, or any power of a mixt number, is ſtill a mixt number, and never an integer. It follows from this, that *the ſquare root of an integer muſt be an integer or an incommenſurable.* Suppoſe that the integer propoſed is B, and that the ſquare root of it is leſs than $a+1$, but greater than a, than it muſt be an incommenſurable; for if it is a commenſurable, let it be $a+\frac{m}{n}$ where $\frac{m}{n}$ repreſents any fraction reduced to its leaſt terms; it would follow that $a + \frac{m}{n}$ ſquared would give an integer number B, the contrary of which we have demonſtrated.

§ 108. It follows from the laſt article, that *the ſquare roots of all numbers but of* 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, &c. (which are the ſquares of the integer numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, *&c.) are incommenſurables*; after the ſame manner, *the cube roots of all numbers but of the cubes of* 1, 2, 3, 4, 5, 6, 7, 8, 9, *&c.* are *incommenſurables:* and quantities that are to one another in the proportion of ſuch numbers muſt alſo have their ſquare roots or cube roots incommenſurable.

§ 109. The roots of ſuch numbers being incommenſurable are expreſſed therefore by placing the proper radical ſign over them; thus, $\sqrt[2]{2}, \sqrt[2]{3}, \sqrt[2]{5}, \sqrt[2]{6}, \sqrt{7}, \sqrt{8}, \sqrt{10}$, *&c.* expreſs numbers incommenſurable with unit. Theſe numbers, though they are incommenſurable themſelves with unit, are *commenſurable in power* with it, becauſe their powers are integers, that is, multiples of unit. They may alſo be commenſurable ſometimes with one another, as the $\sqrt[2]{8}$, and the $\sqrt[2]{2}$, becauſe they are to one another as 2 to 1: And when they have a common meaſure, as $\sqrt[2]{2}$ is the common meaſure of both, then their ratio is reduced to an expreſſion in the leaſt terms, as that of commenſurable quantities, by dividing them by their greateſt common meaſure. This common meaſure is found as in commenſurable quantities, only the root of the common meaſure is to be made their common diviſor. Thus $\frac{\sqrt{12}}{\sqrt{3}}=\sqrt{4}=2$, and $\frac{\sqrt{18a}}{\sqrt{2}}=3\sqrt{a}$.

§ 110. A rational quantity may be reduced to the form of any given ſurd, by raiſing the quantity to the power that is denominated by the name of the ſurd, and then ſetting the radical ſign over it thus, $a=\sqrt[2]{a^2}=\sqrt[3]{a^3}=\sqrt[4]{a^4}=\sqrt[5]{a^5}=\sqrt[n]{a^n}$, and $4=\sqrt{16}=\sqrt[3]{64}=\sqrt[4]{256}=\sqrt[5]{1024}=\sqrt[n]{4^n}$.

§ 111. As ſurds may be conſidered as powers with fractional exponents, *they are reduced to others of the ſame value that ſhall have the ſame radical ſign, by reducing theſe fractional exponents to fractions having the ſame value and a common denominator.* Thus $\sqrt[n]{a}=a^{\frac{1}{n}}$, and $\sqrt[m]{a}=a^{\frac{1}{m}}$, and $\frac{1}{n}=\frac{m}{nm}$, $\frac{1}{m}=\frac{n}{nm}$, and therefore $\sqrt[n]{a}$ and $\sqrt[m]{a}$, reduced to the ſame radical ſign, become $\sqrt[nm]{a^m}$ and $\sqrt[nm]{a^n}$. If you are to reduce $\sqrt[2]{3}$ and $\sqrt[3]{2}$ to the ſame denominator, conſider $\sqrt[2]{3}$ as equal to $3^{\frac{1}{2}}$, the $\sqrt[3]{2}$ as equal to $2^{\frac{1}{3}}$, whoſe indices reduced to a common denominator, you have $3^{\frac{1}{2}} = 3^{\frac{3}{6}}$ and $2^{\frac{1}{3}} = 2^{\frac{2}{6}}$, and conſequently $\sqrt[2]{3} = \sqrt[6]{3^3} = \sqrt[6]{27}$, and $\sqrt[3]{2} = \sqrt[6]{2^2} = \sqrt[6]{4}$; ſo that the propoſed ſurds $\sqrt[2]{3}$ and $\sqrt[3]{2}$ are reduced to other equal ſurds $\sqrt[6]{27}$ and $\sqrt[6]{4}$, having a common radical ſign.

§ 112. *Surds of the ſame rational quantity are multiplied by adding their exponents, and divided by ſubtracting them.*

Thus $\sqrt[2]{a} \times \sqrt[3]{a} = a^{\frac{1}{2}} \times a^{\frac{1}{3}} = a^{\frac{3+2}{6}} = a^{\frac{5}{6}} = \sqrt[6]{a^5}$; and $\dfrac{\sqrt[3]{a}}{\sqrt[5]{a}} = \dfrac{a^{\frac{1}{3}}}{a^{\frac{1}{5}}} = a^{\frac{1}{3} - \frac{1}{5}} = a^{\frac{5-3}{15}} = a^{\frac{2}{15}} = \sqrt[15]{a^2}$; $\sqrt[m]{a} \times \sqrt[n]{a} = a^{\frac{m+n}{mn}}$; $\dfrac{\sqrt[m]{a}}{\sqrt[n]{a}} = a^{\frac{n-m}{mn}}$; $\sqrt[2]{2} \times \sqrt[3]{2} = \sqrt[6]{2^5} = \sqrt[6]{32}$; $\dfrac{\sqrt[2]{2}}{\sqrt[3]{2}} = \sqrt[6]{2}$.

§ 113. If the ſurds are of different rational quantities, as $\sqrt[n]{a^2}$ and $\sqrt[n]{b^3}$, and have the ſame ſign, *multiply theſe rational quantities into one another*

another, or divide them by one another, and ſet the commom radical ſign over their product or quotient. Thus $\sqrt[n]{a^2} \times \sqrt[n]{b^3} = \sqrt[n]{a^2b^3}$; $\sqrt[2]{2} \times \sqrt[2]{5} = \sqrt[2]{10}$; $\dfrac{\sqrt[m]{a^4}}{\sqrt[m]{b^3a}} = \sqrt[m]{\dfrac{a^4}{b^3a}} = \sqrt[m]{\dfrac{a^3}{b^3}}$;

$$\frac{\sqrt[3]{9}}{\sqrt[3]{24}} = \sqrt[3]{\frac{9}{24}} = \sqrt[3]{\frac{3}{8}} = \tfrac{1}{2}\sqrt[3]{3}.$$

If the ſurds have not the ſame radical ſign, *reduce them by the* 11*th* Art. *to ſuch as ſhall have the ſame radical ſign, and proceed as before.* Thus

$$\sqrt[m]{a} \times \sqrt[n]{b} = \sqrt[nm]{a^nb^m};\ \frac{\sqrt[m]{a}}{\sqrt[n]{x}} = \sqrt[mn]{\frac{a^n}{x^m}};\ \sqrt[2]{2} \times \sqrt[3]{4} =$$

$$2^{\frac{1}{2}} \times 4^{\frac{1}{3}} = 2^{\frac{3}{6}} \times 4^{\frac{2}{6}} = \sqrt[6]{2^3 \times 4^2} = \sqrt[6]{8 \times 16}$$

$$= \sqrt[6]{128};\ \frac{\sqrt[3]{4}}{\sqrt[2]{2}} = \frac{4^{\frac{1}{3}}}{2^{\frac{1}{2}}} = \frac{4^{\frac{2}{6}}}{2^{\frac{3}{6}}} = \sqrt[6]{\frac{4^2}{2^3}} = \sqrt[6]{\frac{16}{8}}$$

$= \sqrt[6]{2}$. If the ſurds have any rational coefficients, their product or quotient muſt be prefixed. Thus $2\sqrt[2]{3} \times 5\sqrt[2]{6} = 10\sqrt[2]{18}$.

§ 114. The powers of ſurds are found as the powers of other quantities, *by multiplying their exponents by the index of the power required.* Thus the ſquare of $\sqrt[3]{2}$ is $2^{\frac{1}{3}\times 2} = 2^{\frac{2}{3}} = \sqrt[3]{4}$; the cube of $\sqrt[2]{5} = 5^{\frac{1}{2}\times 3} = 5^{\frac{3}{2}} = \sqrt[2]{125}$. Or you need only, in involving ſurds, *raiſe the quantity under the radical ſign to the power required, continuing the ſame radical ſign; unleſs the index of that power is equal to the name of the ſurd, or a multiple of it, and in that caſe the power of the ſurd becomes rational.* Evolution is performed *by dividing the fraction which is the exponent of the ſurd by the name of the root required.*

Thus the ſquare root of $\sqrt[3]{a^4}$ is $\sqrt[3]{a^2}$, or $\sqrt[6]{a^4}$.

§ 115. The ſurd $\sqrt[m]{a^m x} = a\sqrt[m]{x}$; and in like manner, if a power of any quantity of the ſame name with the ſurd divides the quantity under the radical ſign without a remainder, as here a^m divides $a^m x$, and 25 the ſquare of 5 divides 75, the quantity under the ſign in $\sqrt[2]{75}$, without a remainder, then place the root of that power rationally before the ſign, and the quotient under the ſign, and thus the ſurd will be reduced to a more ſimple expreſſion. Thus $\sqrt[2]{75} = 5\sqrt{3}$; $\sqrt[2]{48} = \sqrt[2]{3 \times 16} = 4\sqrt[2]{3}$; $\sqrt[3]{81} = \sqrt[3]{27 \times 3} = 3\sqrt[3]{3}$.

§ 116. When ſurds by the laſt article are reduced to their leaſt expreſſions, if they have the ſame irrational part, they are added or ſubtracted,

tracted, *by adding or ſubtracting their rational coefficients, and prefixing the ſum or difference to the common irrational part.*

Thus $\sqrt[2]{75} + \sqrt[2]{48} = 5\sqrt{3} + 4\sqrt{3} = 9\sqrt{3}$; $\sqrt[3]{81} + \sqrt[3]{24} = 3\sqrt[3]{3} + 2\sqrt[3]{3} = 5\sqrt[3]{3}$; $\sqrt[2]{150} - \sqrt[2]{54} = 5\sqrt{6} - 3\sqrt{6} = 2\sqrt{6}$; $\sqrt{a^2x} + \sqrt{b^2x} = a\sqrt{x} + b\sqrt{x} = \overline{a+b} \times \sqrt{x}$.

§ 117. *Compound ſurds* are ſuch as conſiſt of two or more joined together. The ſimple ſurds are commenſurable in power, and by being multiplied into themſelves give at length rational quantities; yet compound ſurds multiplied into themſelves commonly give ſtill irrational products. But when any compound ſurd is propoſed, *there is another compound ſurd which multiplied into it gives a rational product.* Thus $\sqrt{a} + \sqrt{b}$ multiplied by $\sqrt{a} - \sqrt{b}$ gives $a - b$, and *the inveſtigation of that ſurd which multiplied into the propoſed ſurd will give a rational product,* is made eaſy by the following Theorems.

THEOREM I.

§ 118. Generally, if you multiply $a^m - b^m$ by $a^{n-m} + a^{n-2m}b^m + a^{n-3m}b^{2m} + a^{n-4m}b^{3m}$, &c. continued till the terms be in number equal to $\frac{n}{m}$, the product ſhall be $a^n - b^n$; for

a^{n-m}

$$a^{n-m}+a^{n-2m}b^{m}+a^{n-3m}b^{2m}+a^{n-4m}b^{3m}, \&c. \ldots b^{n-m}$$
$$\times\,\overline{a^{m}-b^{m}}$$

$$\begin{array}{l} a^{n}+a^{n-m}b^{m}+a^{n-2m}b^{2m}+a^{n-3m}b^{3m}, \&c. \\ \quad -a^{n-m}b^{m}-a^{n-2m}b^{2m}-a^{n-3m}b^{3m}, \&c. -b^{n} \\ \hline a^{n} \quad * \quad * \quad * \quad -b^{n} \end{array}$$

THEOREM II.

$a^{n-m}-a^{n-2m}b^{m}+a^{n-3m}b^{2m}-a^{n-4m}b^{3m}$, &c, multiplied by $\overline{a^{m}+b^{m}}$, gives $a^{n}\mp b^{n}$, which is demonſtrated as the other. Here the ſign of b^{n} is poſitive, when $\frac{n}{m}$ is an odd number.

§ 119. When any binomial ſurd is propoſed, *ſuppoſe the index of each number equal to m, and let n be the leaſt integer number that is meaſured by m, then ſhall* $a^{n-m}\pm a^{n-2m}b^{m}+a^{n-3m}b^{2m}$, &c. *give a compound ſurd, which multiplied into the propoſed ſurd* $a^{m}\mp b^{m}$ *will give a rational product.* Thus to find the ſurd which multiplied by $\sqrt[3]{a}-\sqrt[3]{b}$, will give a rational quantity. Here $m=\frac{1}{3}$, and the leaſt number which is meaſured by $\frac{1}{3}$ is unit; let $n=1$, then ſhall $a^{n-m}+a^{n-2m}b^{m}+a^{n-3m}b^{2m}$, &c. $=a^{1-\frac{1}{3}}+a^{1-\frac{2}{3}}b^{\frac{1}{3}}+a^{0}b^{\frac{2}{3}}=a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}=\sqrt[3]{a^{2}}+\sqrt[3]{ab}+\sqrt[3]{b^{2}}$, which multiplied by $\overline{\sqrt[3]{a}-\sqrt[3]{b}}$, gives $a-b$.

To find the ſurd which multiplied by $\sqrt[4]{a^3}+\sqrt[4]{b^3}=a^{\frac{3}{4}}+b^{\frac{3}{4}}$, gives a rational product. Here $m=\frac{3}{4}$ and $n=3$, and $a^{n-m}-a^{n-2m}b^m+a^{n-3m}b^{2m}$, &c. $=a^{3-\frac{3}{4}}-a^{3-\frac{6}{4}}b^{\frac{3}{4}}+a^{3-\frac{9}{4}}b^{\frac{3}{2}}-a^{3-3}b^{\frac{9}{4}}=a^{\frac{9}{4}}-a^{\frac{6}{4}}b^{\frac{3}{4}}+a^{\frac{3}{4}}b^{\frac{6}{4}}-b^{\frac{9}{4}}=\sqrt[4]{a^9}-\sqrt[4]{a^6b^3}+\sqrt[4]{a^3b^6}-\sqrt[4]{b^9}$.

THEOREM III.

§ 120. Let $a^m\pm b^l$ be multiplied by $a^{n-m}\mp a^{n-2m}b^l+a^{n-3m}b^{2l}\mp a^{n-4m}b^{3l}+$ *&c.* and the product ſhall give $a^n\pm b^{\frac{n}{m}l}$: *therefore n muſt be taken the leaſt integer that ſhall give $\frac{nl}{m}$ alſo an integer.*

Dem. $a^{n-m}\mp a^{n-2m}b^l+a^{n-3m}b^{2l}\mp a^{n-4m}b^{3l}\ldots$:

$$\begin{array}{l}
\times\,\overline{a^m\pm b^l} \qquad\qquad [\ldots\text{&c. } a^0 b^{\overline{\frac{n}{m}-1\times l}} \\
\hline
a^n\mp a^{n-m}b^l+a^{n-2m}b^{2l},\ \text{&c.} \\
\quad\pm a^{n-m}b^l-a^{n-2m}b^{2l},\ \text{&c.}\ \pm b^{\frac{n}{m}l} \\
\hline
a^n \quad * \quad * \quad * \quad \pm b^{\frac{n}{m}l}.
\end{array}$$

The ſign of $b^{\frac{nl}{m}}$ is poſitive only when $\frac{n}{m}$ is an odd number, and the binomial propoſed is a^m+b^l.

§ 121. If any binomial ſurd is propoſed whoſe two numbers have different indices, let theſe be m and l, and take n equal to the leaſt integer number that is meaſured by m and by $\frac{m}{l}$; and $a^{n-m} \mp a^{n-2m}b^{l} + a^{n-3m}b^{2l} \mp a^{n-4m}b^{3l}$, &c. ſhall give a compound ſurd, which multiplied by the propoſed $a^{m} \pm b^{l}$ ſhall give a rational product. Thus $\sqrt[2]{a} - \sqrt[2]{b}$ being given, ſuppoſe $m = \frac{1}{2}$, $l = \frac{1}{3}$, and $\frac{m}{l} = \frac{3}{2}$, therefore you have $n = 3$, and $a^{n-m} + a^{n-2m}b^{l} + a^{n-3m}b^{2l} + a^{n-4m}b^{3l} + \&c. = a^{3-\frac{1}{2}} + a^{3-1}b^{\frac{1}{3}} + a^{3-\frac{3}{2}}b^{\frac{2}{3}} + a^{3-2}b + a^{3-\frac{5}{2}}b^{\frac{4}{3}} + a^{0}b^{\frac{5}{3}} = a^{\frac{5}{2}} + a^{2}b^{\frac{1}{3}} + a^{\frac{3}{2}}b^{\frac{2}{3}} + ab + a^{\frac{1}{2}}b^{\frac{4}{3}} + b^{\frac{5}{3}} = \sqrt{a^{5}} + a^{2} \times \sqrt[3]{b} + \sqrt[2]{a^{3}} \times \sqrt[3]{b^{2}} + ab + \sqrt[2]{a} \times \sqrt[3]{b^{4}} + \sqrt[3]{b^{5}} = a^{2}\sqrt{a} + a^{2} \times \sqrt[3]{b} + a\sqrt{a} \times \sqrt[3]{b^{2}} + ab + b\sqrt{a} \times \sqrt[3]{b} + b \times \sqrt[3]{b^{2}}$, which multiplied by the $\sqrt[2]{a} - \sqrt[3]{b}$, gives $a^{n} - b^{\frac{nl}{m}} = a^{3} - b^{2}$.

§ 122. By theſe Theorems any binomial ſurd whatſoever being given, you may find a ſurd which multiplied by it ſhall give a rational product.

Suppoſe that a binomial ſurd was to be divided by another, as $\sqrt[2]{20} + \sqrt[2]{12}$, by $\sqrt[2]{5} - \sqrt[2]{3}$, the quotient may be expreſſed by $\frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}}$. But it may be expreſſed in a more ſimple form by *multiplying both numerator and denominator by that ſurd which multiplied into the denominator gives a rational product.* Thus $\frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}}=$

$$\frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}}\times\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}=\frac{\sqrt{100}+2\sqrt{60}+6}{5-3}$$

$$=\frac{16+2\sqrt{60}}{2}=8+2\sqrt{15}.$$

§ 123. In general, when any quantity is divided by a binomial ſurd, as $a^m \pm b^l$, where m and l repreſent any fractions whatſoever, *take* n *the leaſt integer number that is meaſured by* m *and* $\frac{m}{l}$, *multiply both numerator and denominator by* $a^{n-m}+a^{n-2m}b^l+a^{n-3m}b^{2l}$, &c. *and the denominator of the product will become rational, and equal to* $a^n - b^{\frac{nl}{m}}$; *then divide all the members of the numerator by this rational quantity, and the quote ariſing will be that of the propoſed quantity divided by the binomial ſurd, expreſſed in its leaſt terms.*

Thus $\frac{3}{\sqrt{5}-\sqrt{2}}=\frac{3\sqrt{5}+3\sqrt{2}}{3}=\sqrt{5}+\sqrt{2}$;

$$\frac{\sqrt{6}}{\sqrt{7}-\sqrt{3}}=\frac{\sqrt{42}+\sqrt{18}}{4};\ \frac{\sqrt[3]{20}}{\sqrt[3]{4}-\sqrt[3]{2}}=\frac{\sqrt[3]{20}}{\sqrt[3]{4}-\sqrt[3]{2}}$$

$$\times\frac{\sqrt[3]{16}+2+\sqrt[3]{4}}{\sqrt[3]{16}+2+\sqrt[3]{4}}=\frac{\sqrt[3]{20}}{\sqrt[3]{4}-\sqrt[3]{2}}\times\frac{2\sqrt[3]{2}+2+\sqrt[3]{4}}{2\sqrt[3]{2}+2+\sqrt[3]{4}}$$

=

$$=\frac{2\sqrt[3]{40}+2\sqrt[3]{20}+\sqrt[3]{80}}{2}=2\sqrt[3]{5}+\sqrt[3]{20}+\sqrt[3]{10}:$$

Alſo $\frac{\sqrt[2]{10}}{\sqrt[2]{2}-\sqrt[3]{3}}$ = (becauſe $m=\frac{1}{2}$, $l=\frac{1}{3}$, $n=3$, and $a^n-b^{\frac{nl}{m}}=8-9=-1$) =

$$\frac{4\sqrt{20}+4\sqrt{10}\times\sqrt[3]{3}+2\sqrt{20}\times\sqrt[3]{9}+6\sqrt{10}+3\sqrt{20}\times\sqrt[3]{3}+3\sqrt{10}\times\sqrt[3]{9}}{-1}$$

$$=-8\sqrt{5}-4\sqrt{10}\times\sqrt[3]{3}-8\sqrt{5}\times\sqrt[3]{9}-6\sqrt{10}-6\times\sqrt{5}\times\sqrt[3]{3}-3\sqrt{10}\times\sqrt[3]{9}.$$

§ 124. When the ſquare root of a ſurd is required, it may be found nearly *by extracting the root of a rational quantity that approximates to its value.* Thus to find the ſquare root of $3+2\sqrt{2}$, we firſt calculate $\sqrt{2}=1,41421$, and therefore $3+2\sqrt{2}=5,82842$, whoſe root is found to be nearly 2, 41421: ſo that $\sqrt[2]{3+2\sqrt{2}}$ is nearly 2, 41421. But ſometimes we may be able to expreſs the roots of ſurds exactly by other ſurds; as in this example the ſquare root of $3+2\sqrt{2}$ is $\overline{1+\sqrt{2}}$, for $\overline{1+\sqrt{2}}\times\overline{1+\sqrt{2}}=1+2\sqrt{2}+2=3+2\sqrt{2}$.

In order to know when and how this may be found, let us ſuppoſe that $x+y$ is a binomial ſurd, whoſe ſquare will be x^2+y^2+2xy: If x and y are quadratic ſurds, then x^2+y^2 will be rational, and $2xy$ irrational; ſo that $2xy$

ſhall always be leſs than x^2+y^2, becauſe the difference is $x^2+y^2-2xy=\overline{x-y}^2$, which is always poſitive. Suppoſe that a propoſed ſurd conſiſting of a rational part A, and an irrational part B, coincides with this, then $x^2+y^2=A$ and $xy=\frac{1}{2}B$: Therefore by what was ſaid of Equations, *Chap.* 13th,

$$y^2=A-x^2=\frac{B^2}{4x^2}, \text{ and therefore}$$

$$Ax^2-x^4=\frac{B^2}{4}, \text{ and } x^4-Ax^2+\frac{B^2}{4}=0;$$

from whence we have $x^2=\frac{A+\sqrt{A^2-B^2}}{2}$ and $y^2=\frac{A-\sqrt{A^2-B^2}}{2}$. Therefore when a quantity partly rational and partly irrational is propoſed to have its root extracted, *call the rational part* A, *the irrational* B, *and the ſquare of the greateſt member of the root ſhall be* $\frac{A+\sqrt{A^2-B^2}}{2}$, *and the ſquare of the leſſer part ſhall be* $\frac{A-\sqrt{A^2-B^2}}{2}$. And as often as the ſquare root of A^2-B^2 can be extracted, the ſquare root of the propoſed binomial ſurd may be expreſſed itſelf as a binomial ſurd. For example, if $3+2\sqrt{2}$ is propoſed, then $A=3$, $B=2\sqrt{2}$, and $A^2-B^2=9-8=1$. Therefore $x^2=\frac{A+\sqrt{A^2-B^2}}{2}=2$, and $y^2=\frac{A-\sqrt{A^2-B^2}}{2}=1$. Therefore $x+y=1+\sqrt{2}$.

To find the ſquare root of $-1\sqrt{-8}$, ſuppoſe $A=-1$, $B=\sqrt{-8}$, ſo that $A^2-B^2=9$, and $\frac{A+\sqrt{A^2-B^2}}{2}=\frac{-1+3}{2}=1$, and $\frac{A-\sqrt{A^2-B^2}}{2}=\frac{-1-3}{2}=-2$, therefore the root required is $1+\sqrt{-2}$.

§ 125. But though x and y are not quadratic ſurds or roots of integers, if they are the roots of like ſurds, as if they are equal to $\sqrt{m\sqrt{z}}$ and $\sqrt{n\sqrt{z}}$, where m and n are integers, then $A=\overline{m+n}\times\sqrt{z}$ and $\frac{1}{2}B=\sqrt[2]{mnz}$; $A^2-B^2=\overline{m-n}|^2\times z$, and $x^2=\frac{A+\sqrt{A^2-B^2}}{2}=\frac{\overline{m+n}\times z+\overline{m-n}|^2\times z}{2}=m\sqrt{z}$, $y^2=\frac{A-\sqrt{A^2-B^2}}{2}=n\sqrt{z}$, and $x+y=\sqrt{m\sqrt{z}}+\sqrt{n\sqrt{z}}$. The part A here eaſily diſtinguiſhes itſelf from B by its being greater.

§ 126. If x and y are equal to $\sqrt{m\sqrt{z}}$ and $\sqrt{n\sqrt{t}}$, then $x^2+2xy+y^2=m\sqrt{z}+n\sqrt{t}+2\sqrt{mn\sqrt{zt}}$. So that if z or t be not multiples one of the other, or of ſome number that meaſures them both by a ſquare number, then will A itſelf be a binomial.

§ 127. Let $x+y+z$ expreſs any trinomial ſurd, its ſquare $x^2+y^2+z^2+2xy+2xz+2yz$ may be ſuppoſed equal to $A+B$ as before. But rather multiply any two radicals as $2xy$ by $2xz$, and divide by the third $2yz$, which gives the quotient $2x^2$ rational, and double the ſquare of the ſurd x required. The ſame rule ſerves when there are four quantities, $\overline{x^2+y^2+z^2+s^2+}$ $2xy+2xs+2xz+2yz+2ys+2zs$, multiply $2xy$ by $2xs$, and the product $4x^2sy$ divided by $2sy$ gives $2x^2$ a rational quotient, half the ſquare of $2x$. In like manner $2xy\times 2yz=4y^2xz$, which divided by $2xz$ another member gives $2y^2$, a rational quote, the half of the ſquare of $2y$. In the ſame manner z and s may be found; and their ſum $x+y+z+s$, the ſquare root of the ſeptinomial $\overline{x^2+y^2+z^2+s^2}+2xy+2xs+$ $2xz+2yz+2ys$, diſcovered.

For example, to find the ſquare root of $10+\sqrt{24}+\sqrt{40}+\sqrt{60}$; I try $\frac{\sqrt{24}\times\sqrt{40}}{\sqrt{60}}$, which I find to be $\sqrt{16}=4$, the half of the ſquare root of the double of which, *viz.* $\frac{1}{2}\times\sqrt{8}=\sqrt{2}$, is one member of the ſquare root required; next $\frac{\sqrt{24}\times\sqrt{60}}{\sqrt{40}}=6$, the half of the ſquare root of the double of which is $\sqrt{3}$, another member of the root required; laſtly, $\frac{\sqrt{40}\times\sqrt{60}}{\sqrt{24}}=10$, which gives $\sqrt{5}$ for the third member of the root required: From which we conclude that the ſquare root of $10+\sqrt{24}+\sqrt{40}+60$ is $\sqrt{2}+\sqrt{3}+\sqrt{5}$; and trying you find it ſucceeds, ſince multiplied by itſelf it gives the propoſed quadrinomial.

§ 127 For extracting the higher roots of a binomial, whoſe two members being ſquared are commenſurable numbers, there is the following

RULE.

* "*Let the quantity be* $A \pm B$, *whereof* A *is the greater part, and c the exponent of the root required. Seek the leaſt number n whoſe power* n^c *is diviſible by* $AA - BB$, *the quotient being* Q. *Compute* $\sqrt[c]{\overline{A+B} \times \sqrt{Q}}$ *in the neareſt integer number, which ſuppoſe to be r. Divide* $A\sqrt{Q}$ *by its greateſt rational diviſor, and let the quotient be s, and let* $\frac{r+\frac{n}{r}}{2s}$, *in the neareſt integer number, be t, ſo ſhall the root required be* $\frac{ts \pm \sqrt{t^2s^2 - n}}{\sqrt[2c]{Q}}$, *if the c root of* $A \pm B$ *can be extracted.*

EXAMPLE I.

Thus to find the cube root of $\sqrt{968} + 25$, we have $A^2 - B^2 = 343$, whoſe diviſors are 7, 7, 7, whence $n = 7$, and $Q = 1$. Further, $\overline{A+B} \times \sqrt{Q}$, that is, $\sqrt{968} + 25$ is a little more than 56, whoſe neareſt cube root is 4. Wherefore $r=4$. Again, dividing $\sqrt{968}$ by its greateſt rational diviſor, we have $A\sqrt{Q}=22\sqrt{2}$, and the radical part $\sqrt{2}=s$, and $\frac{r+\frac{n}{r}}{2s}$, or $\frac{5}{2\sqrt{2}}$, in the neareſt integers, is $2=t$. And laſtly, $ts=2\sqrt{2}$, $\sqrt{t^2s^2-n}=1$, and $\sqrt[2c]{Q}=\sqrt[6]{1}=1$. Whence $2\sqrt{2}+1$ is the root, whoſe cube, upon trial, I find to be $\sqrt{968}+25$.

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* Arithm. Univerſal. p. 59.

EXAMPLE II.

To find the cube root of $68-\sqrt{4374}$; we have $A^2-B^2=250$, whoſe diviſors are 5, 5, 5, 2. Thence $n=5\times 2=10$, and $Q=4$, and $\sqrt[c]{\overline{A+B}\times\sqrt{Q}}$ or $\sqrt[3]{\overline{68+\sqrt{4374}}\times 2}$ is nearly $7=r$; again, $A\sqrt{Q}$, or $68\times\sqrt{4}=136\times\sqrt{1}$, that is, $s=1$, and $\frac{r+\frac{n}{r}}{2s}$, or $\frac{7+\frac{10}{7}}{2}$, is nearly $=4=t$. Therefore $ts=4$, $\sqrt{t^2s^2-n}=\sqrt{6}$, and $\sqrt[2c]{Q}=\sqrt[6]{4}=\sqrt[3]{2}$, whence the root to be tried is $\frac{4-\sqrt{6}}{\sqrt[3]{2}}$.

EXAMPLE III.

Suppoſe the fifth root of $29\sqrt{6}+41\sqrt{3}$ is demanded, $A^2-B^2=3$, and $n=3$; $Q=81$,

$r=5$, $s=\sqrt{6}$, $t=1$, $ts=\sqrt{6}$, $\sqrt{t^2s^2-n}=\sqrt{3}$, and $\sqrt[2c]{Q}=\sqrt[10]{81}=\sqrt[5]{9}$. And therefore trial is to be made with $\frac{\sqrt{6}+\sqrt{3}}{\sqrt[5]{9}}$.

In theſe operations, if the quantity is a fraction, or if its parts have a common diviſor, you are to extract the root of the *numerator* and *denominator*, or of the *factors* ſeparately. Thus to extract the cube root of $\sqrt{242}-12\frac{1}{2}$, this reduced to a common denominator is $\frac{\sqrt{968}-25}{2}$. And the roots of the numerator and denominator, ſeparately found, give the root $\frac{2\sqrt{2}-1}{\sqrt[3]{2}}$. And if you ſeek any root of $\sqrt[3]{3993}+\sqrt[6]{17578125}$, divide its parts by the common diviſor $\sqrt[3]{3}$, and the quotient being $11+\sqrt{125}$, the root of the quantity propoſed will be found by taking the roots of $\sqrt[3]{3}$ and of $11+\sqrt{125}$, and multiplying them into each other.

§ 128. The ground of this Rule may be explained from the following

THEOREM

Let the ſum or difference of two quantities x and y be raiſed to a power whoſe exponent is c, *and let the 1ſt, 3d, 5th, 7th, &c. terms of that power, collected into one ſum, be called* A, *and the reſt of the terms, in the even places, call* B; *the difference of the ſquares of* A *and* B *ſhall be equal to the difference of the ſquares of x and y raiſed to the ſame power c.*

For the terms in the c power of $x+y$ (writing for their coefficients, reſpectively, 1, c, d, e, &c. are
$x^c + cx^{c-1}y + dx^{c-2}y^2 + ex^{c-3}y^3 +$, &c. $= A + B$,
and the ſame power of $x-y$ (changing the ſigns in the even places) is
$x^c - cx^{c-1}y + dx^{c-2}y^2 - ex^{c-3}y^3 +$, &c. $= A - B$,
and therefore $\overline{x+y}|^c \times \overline{x-y}|^c = \overline{A+B} \times \overline{A-B}$
$= A^2 - B^2 \; (= \overline{\overline{x+y} \times \overline{x-y}}|^c) = \overline{x^2-y^2}|^c$.
Q. E. D.

Let one, or both, of the quantities x, y, be a quadratic *ſurd*, that is, let $x+y$, the c root of the propoſed binomial $A+B$ belong to one of theſe forms, $p+l\sqrt{q}$, $k\sqrt{p}+q$, or $k\sqrt{p}+l\sqrt{q}$. And it follows,

1. If $x+y=p+l\sqrt{q}$, that, c being any whole number, A, the ſum of the *odd* terms, will be a rational number; and B, the ſum of the terms in the *even* places, each of which involves an *odd* power of y will be a rational number multiplied into the quadratic ſurd $\sqrt{q}$.

2. Let c, the exponent of the root ſought, be an odd number, as we may always ſuppoſe

it, becauſe if it is even, it may be halved by the extraction of the ſquare root, till it becomes odd; and let $x+y=k\surd p+q$. Then A will involve the ſurd $\surd p$, and B will be rational.

3. But if both members of the root are irrational $(x+y=k\surd p+l\surd q)$ A and B are both irrational, the one involving $\surd p$, and the other the ſurd $\surd q$.

And in all theſe caſes, it is eaſily ſeen that when x is greater than y, A will be greater than B.

§ 129. From this compoſition of the binomial A + B, we are led to its reſolution, as in the foregoing rule, by theſe ſteps.

I.

When A is *rational*, and A^2-B^2 is a perfect c power.

1. By the *Theorem*, $A^2-B^2=\overline{x^2-y^2}|^c$ *accurately*; and therefore extracting the c root of A^2-B^2 it will be x^2-y^2. Call this root n.

2. Extract in the neareſt integer, the c root of A + B, it will be (*nearly*) $x+y$. Which put $=r$.

3. Divide x^2-y^2 $(=n)$ by $x+y$ $(=r)$ the quotient is (*nearly*) $x-y$; and the ſum of the diviſor and quotient is (*more nearly*) $2x$; that is, if an integer value of x is to be found, it will be the neareſt to $\frac{r+\frac{n}{r}}{2}$.

4. $x^2 - \overline{x^2 - y^2} = y^2$; or, $\overline{\frac{r+\frac{n}{r}}{2}}\Big|^2 - n = y^2$: whence $y = \sqrt{\overline{\frac{r+\frac{n}{r}}{2}}\Big|^2} - n$, and therefore, putting $t = \frac{r+\frac{n}{r}}{2}$, the root ſought $x + y = t + \sqrt{t^2 - n}$; the ſame expreſſion as in the rule, when $Q = 1$, $s = 1$, that is, when $A^2 - B^2$ is a perfect c power, and the greater member A is rational.

II.

When A is *irrational*, and $Q = 1$.

By the ſame proceſs, $x = \frac{r+\frac{n}{r}}{2}$ $(= T)$ and $y = \sqrt{T^2 - n}$. But ſeeing A is ſuppoſed irrational, and c an odd number, x will be irrational likewiſe; and they will both involve the ſame irreducible ſurd $\sqrt{p}$, or s, which is found by dividing A by its greateſt rational diviſor. Write therefore for x or T, its value $t \times s$, and $x + y = ts + \sqrt{t^2 s^2 - n}$.

III.

If the c root of $A^2 - B^2$ cannot be taken, multiply $A^2 - B^2$ by a number Q, ſuch as that the product may be the (leaſt) perfect c power n^c $(= A^2 Q - B^2 Q.)$ And now (inſtead of

A +

$A + B$) extract the c root of $\overline{A+B} \times \sqrt{Q}$, which, found as above, will be $ts + \sqrt{t^2s^2 - n}$; and conſequently the c root of $A + B$ will be $ts + \sqrt{t^2s^2 - n}$, divided by the c root of $\sqrt{Q}$; that is, $\dfrac{ts + \sqrt{t^2s^2 - n}}{\sqrt[2c]{Q}}$.

It is required in the rule that *a perfect c power* (n^c) *be found which ſhall be a multiple of* $A^2 - B^2$ *by the whole number* Q. To find this power, let the given number $A^2 - B^2$ be repreſented by the product $a^m b^p d f$; whoſe ſingle diviſors let be $a, a, a, \ldots . b, b, b, \ldots . d, f$; and the product of theſe diviſors raiſed to the power c, which is $a^c b^c d^c f^c$, divided by $a^m b^p d f$ will give the quotient $a^{c-m} b^{c-p} d^{c-1} f^{c-1} = Q$ a whole number, provided ſome index, as m or p, be not greater than c. If it is, take, inſtead of the ſingle diviſor a or b, a^2 or b^2, a^3 or b^3, *&c.* till there be no negative index in the quotient; that is, till Q be a whole number.

§ 130. We may add the following remarks

1. If the reſidual $A - B$ is given, it is evident from its geneſis by involution, that the ſame rule gives its root $x - y$.

2. The extracting the c root of $A + B$, or of $\overline{A+B} \times \sqrt{Q}$, in the neareſt integer, neglecting the fractional part, will always give $x + y$ ſuch, that the value of x which reſults in the operation ſhall not differ from its true value by

unity; that is, it ſhall be the true integer value ſought.

For f being ſome proper fraction, let $x+y\pm f$ be the accurate value of $\sqrt[c]{\overline{A+B}\times\sqrt{Q}}$, and let the quotient of x^2-y^2 divided by it be $x-y\mp g$, then, the ſum of the diviſor and quotient being $2x\pm f\mp g$, if our reckoning the fractional part could make a difference of unity in the value of x, it would follow that $f-g$ or $g-f=2$. Which is abſurd, g, as well as f, being a proper fraction.

3. If both A and B are irrational; or, if the leſſer of the two members is rational, no root denominated by an even number can be found.

4. When the greater member is rational, and the exponent c is an even number, it is ambiguous whether the greater member of the root is rational or ſurd. And though a root in the form of $p+l\sqrt{q}$ is not found, yet a root in the form of $k\sqrt{p}+q$, or, that failing, in the form $k\sqrt{p}+l\sqrt{q}$ may be obtained.

If we look for a root $k\sqrt{p}+q$, we are now to ſubtract $x-y$ from $x+y$, and half the remainder will give y (or q) the rational part. And to x^2-y^2 $(=n)$ adding y^2, the ſum will be x^2. So that $y=\dfrac{r-\frac{n}{r}}{2}$, and $x=\sqrt{\left|\dfrac{r-\frac{n}{r}}{2}\right|^2+n}$; the expreſſions being the ſame as when c is odd, with the ſign of n changed. If this does not ſucceed, and a prime number ſtands under the radical ſign, no farther trial need be made.

But if a compoſite number ſtands under the radical ſign, the root may poſſibly belong to the form $k\sqrt{p}+l\sqrt{q}$; and that compoſite number being $p\times q$, ſince $k^2p-l^2q=n$, and $k\sqrt{p}=x$, the numbers k, l, may be ſought for in the neareſt integers, and trial made with $k\sqrt{p}+l\sqrt{q}$; as in this

EXAMPLE.

To find the fourth root of $49849-2895\sqrt{224}$.

The 4th root of A^2-B^2 is $157=x^2-y^2=n$, and the 4th root of $A-B$, that is, $x-y=r=9$ *nearly*: and $\frac{n}{r}=\frac{157}{9}=17$ *nearly*. Whence $x=\frac{9+17}{2}=13$. But now the leaſt radical factor in B being $\sqrt{14}=\sqrt{7\times 2}$, I put $13\ (=x)$ $=k\sqrt{7}$, and k in the neareſt integer $=5$. Again $k^2p-l^2q=n=175-l^2\times 2=157$; that is, $l^2\times 2=18$, and $l=3$; which gives the root $5\sqrt{7}-3\sqrt{2}$.

In this manner the even roots may be ſought immediately. But to avoid ambiguity and needleſs trouble, it is better firſt to depreſs them by extracting the ſquare root, as in § 124.

A SUP-

A SUPPLEMENT TO THIS CHAPTER

§ 131. THERE occur ſometimes, eſpecially in the reſolution of *cubic* equations by *Cardan*'s Rule (*Part.* II. § 79.) binomials of this form $A \pm B\sqrt{-q}$, whoſe cube roots muſt be found. To theſe the foregoing rule cannot be applied throughout, becauſe of the imaginary factor $\sqrt{-q}$. Yet if the root is expreſſible in rational numbers, the firſt ſtep of that rule will often lead us to it in a ſhort way, not merely tentative, the trials being confined to known limits.

For it being, univerſally, $\sqrt[c]{A^2-B^2q}=x^2-y^2$ and, in the preſent caſe, $\sqrt[3]{A^2+B^2q}(=x^2-y^2)$ $=p^2+l^2\times q$; if we divide the part under the radical ſign by its greateſt rational diviſor, the quote is the imaginary ſurd $\sqrt{-q}$, and from $\sqrt[3]{A^2+B^2q}$, ſubtracting p^2, the ſquare of ſome diviſor of A, the remainder is $l^2 \times q$, a known multiple of the ſquare of l a diviſor of B.

That p and l are diviſors of A and B reſpectively is evident; for cubing $p+l\sqrt{-q}$, you find $A=p\times\overline{p^2-3l^2q}$, $B=l\times\overline{3p^2-l^2q}$. And the ſigns of p and l muſt be ſuch as will give the products of $p\times\overline{p^2-3l^2q}$, $l\times\overline{3p^2-l^2q}$ of the ſame ſigns as A and B reſpectively.

EXAMPLE.

To find the cube root of $81+\sqrt{-2700}=81+30\sqrt{-3}$.

Here $A=81$, $B=30$, $q=3$; $\sqrt[3]{81\times81+2700}=21=p^2+l^2q$. Subtracting therefore from 21, the ſquare of $(p)\ \pm 3$, which is a diviſor of A, there remains $(l^2\times q=)\ 2\times2\times3$. And $(l=)\ 2$ is a diviſor of 30. Laſtly, A $(=p\times\overline{p^2-3l^2q})$ being poſitive, and the factor p^2-3l^2q negative, p muſt have the negative ſign; and for the like reaſon $l=+2$. So that the root is $-3+2\sqrt{-3}$.

It will be ſhewn in the ſecond Part of this Treatiſe that "every cube or other power has as many roots, real and imaginary, as there are units in the exponent of the power;" particularly, that *unity* itſelf has the cube roots 1, $\frac{-1+\sqrt{-3}}{2}$, and $\frac{-1-\sqrt{-3}}{2}$. If therefore we would find the other two cube roots, in this example, ſeeing $z^3=z^3\times1$, and $\sqrt[3]{z^3}\times\sqrt[3]{1}=z$

(z^3 repreſenting any cube whatever, and z any of its roots) we are to multiply $-3+2\sqrt{-3}$, the root already found, by $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$, and by $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$, and the products $-\frac{3}{2}-\frac{5}{2}\sqrt{-3}$, and $\frac{9}{2}+\frac{1}{2}\sqrt{-3}$ will be the root required.

Or, becauſe the denominator of the imaginary roots of unity is 2, taking $p=\frac{3}{2}$, one half of a diviſor of A, we have $21-\frac{9}{4}=\frac{75}{4}=\frac{25}{4}\times 3=l^2q$, that is $l=\frac{5}{2}$; and p^2-3l^2q as well as $3p^2-l^2q$ being negative, both p and l muſt be negative, and the root is $-\frac{3}{2}-\frac{5}{2}\sqrt{-3}$. Again take $p=\frac{9}{2}$, and you ſhall find $l=+\frac{1}{2}$; ſo the remaining root is $\frac{9}{2}+\frac{1}{2}\sqrt{-3}$, as before.

We may here obſerve that the operation ought to be abridged, where it can be done, by dividing the given binomial by the greateſt cube that it contains; and finding the root of the quotient; which multiplied by the root of the cube by which you divided, will give the root required. Thus, in the foregoing Example, $81+\sqrt{-2700}=27\times\overline{3+\sqrt{-\frac{100}{27}}}$, and the roots of $3+\sqrt{-\frac{100}{27}}$ being now, more eaſily, found to be $-1+2\sqrt{-\frac{1}{3}}$, $-\frac{1}{2}-\frac{5}{2}\sqrt{-\frac{1}{3}}$, and $\frac{3}{2}+\frac{1}{2}\sqrt{-\frac{1}{3}}$, theſe multiplied by 3, the cube root of 27, give the roots required the ſame as above.

“ If the coefficient of the imaginary member of the binomial has a contrary ſign, the roots will be the ſame, with the ſigns of the imaginary parts changed." Thus the cube roots of $81 - \sqrt{-2700}$, or $81 - 30\sqrt{-3}$, will be $-3 - 2\sqrt{-3}$, $-\frac{3}{2} + \frac{5}{2}\sqrt{-3}$, and $\frac{9}{2} - \frac{1}{2}\sqrt{-3}$. And therefore $\sqrt[3]{81 + \sqrt{-2700}} + \sqrt[3]{81 - \sqrt{-2700}} = -3 \times 2 = -6$, or $= -\frac{3}{2} \times 2 = -3$, or $= \frac{9}{2} \times 2 = 9$; the imaginary parts vaniſhing by the contrariety of their ſigns.

We may obſerve likewiſe, that ſuch roots, whether expreſſible in rational numbers, or not, may be found by evolving the binomial $A + B\sqrt{-q}$ by the *Theorem* in *pag.* 41, and ſumming the alternate terms. As, in the foregoing example, $\overline{81 + 30\sqrt{-3}}|^{\frac{1}{3}}$, or rather $\overline{81}|^{\frac{1}{3}} \times \overline{1 + \frac{10}{27}\sqrt{-3}}|^{\frac{1}{3}}$, being expanded into a ſeries, the ſum of the odd terms will continually approach to $4.5 = \frac{9}{2}$, and the ſum of the coefficients of the even terms to $\frac{1}{2}$, which is the coefficient of the imaginary part. But for a general and elegant ſolution, recourſe muſt be had to Mr. *de Moivre*'s Appendix to Dr. *Saunderſon*'s Algebra, and the continuation of it in *Philoſ. Tranſ.* N° 451. What has been explained above may ſerve, for the preſent, to give the Learne ſome notion of the compoſition and reſolution of thoſe cubes; that he need not hereafter be ſurpriſed to meet with expreſſions of real quantities which involve imaginary roots.

End of the FIRST PART.

A
TREATISE
OF
ALGEBRA.

PART II.
Of the Geneſis and Reſolution of EQUATIONS of all degrees; and of the different Affections of the ROOTS.

CHAP. I.
Of the Geneſis and Reſolution of Equations in general; and the number of roots an equation of any degree may have.

§ 1. AFTER the ſame manner as the higher powers are produced by the multiplication of the lower powers of the ſame root; equations of ſuperior orders are generated by the multiplication of equations of inferior orders involving the ſame unknown quantity. And *an equation of* *any dimenſions may be conſidered as produced by the multiplication of as many ſimple equations as it has dimenſions*; or *of any other equations whatſoever, if the ſum of their dimenſions is equal to the dimenſion of that equation.* Thus any *cubic* equation may be conceived as generated by the multiplication of *three* ſimple equations, or of *one quadratic* and *one ſimple* equation. A *biquadratic* as generated by the multiplication of *four ſimple* equations, or of *two quadratic* equations; or laſtly, of *one cubic* and *one ſimple* equation.

§ 2. If the equations which you ſuppoſe multiplied by one another are *the ſame*, then the equation generated will be nothing elſe but ſome power of thoſe equations, and the operation is merely *involution*; of which we have treated already: and, when any ſuch equation is given, the ſimple equation by whoſe multiplication it is produced is found by *evolution*, or the extraction of a root.

But when the equations that are ſuppoſed to be multiplied by each other are *different*, then other equations than powers are generated; which to reſolve into the ſimple equations whence they are generated, is a different operation from involution, and is what is called, *The reſolution of equations.*

But as evolution is performed by obſerving and tracing back the ſteps of involution; ſo to diſcover the rules for the reſolution of equations, we muſt carefully obſerve their *generation.*

§ 3. Suppoſe the unknown quantity to be x and its values in any ſimple equations to be a, b, c, d, &c. then thoſe ſimple equations, by bringing all the terms to one ſide, become $x-a=0$, $x-b=0$, $x-c=0$, &c. And, the product of any two of theſe, as $\overline{x-a}\times\overline{x-b}=0$ will give a *quadratic* equation, or an equation of two dimenſions. The product of any three of them, as $\overline{x-a}\times\overline{x-b}\times\overline{x-c}=0$ will give a *cubic* equation, or one of three dimenſions. The product of any four of them will give a *biquadratic* equation, or one of four dimenſions, as $\overline{x-a}\times\overline{x-b}\times\overline{x-c}\times\overline{x-d}=0$. And, *in general*, “ *In the equation produced, the higheſt dimenſion of the unknown quantity will be equal to the number of ſimple equations that are multiplied by each other.*

§ 4. When any equation equivalent to this biquadratic $\overline{x-a}\times\overline{x-b}\times\overline{x-c}\times\overline{x-d}=0$ is propoſed to be reſolved, the whole difficulty conſiſts in finding the ſimple equations $\overline{x-a}=0$, $\overline{x-b}=0$, $\overline{x-c}=0$, $\overline{x-d}=0$, by whoſe multiplication it is produced; for each of theſe ſimple equations gives one of the values of x, and one ſolution of the propoſed equation. For, if any of the values of x deduced from thoſe ſimple equations be ſubſtituted in the propoſed equation, in place of x, then all the terms of that equation will vaniſh, and the whole be found equal to nothing. Becauſe when it is ſuppoſed that $x = a$, or $x = b$, or $x = c$, or $x = d$, then the product $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d}$ does vaniſh, becauſe one of the factors is equal to nothing. There are therefore four ſuppoſitions that give $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d} = 0$ according to the propoſed equation; that is, there are four roots of the propoſed equation. And after the ſame manner, "*Any other* equation admits of as many ſolutions as there are ſimple equations multiplied by one another that produce it," or "as many as there are units in the higheſt dimenſion of the unknown quantity in the propoſed equation."

§ 5. But as there are no other quantities whatſoever beſides theſe four (a, b, c, d,) that ſubſtituted in the product $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d}$, in the place of x, will make the product vaniſh; therefore, the equation $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d} = 0$, cannot poſſibly have more than theſe four roots, and cannot admit of more ſolutions than four. If you ſubſtitute in that product a quantity neither equal to a, nor b, nor c, nor d, which ſuppoſe e, then ſince neither $e-a$, $e-b$, $e-c$, nor $e-d$ is equal to nothing; their product $\overline{e-a} \times \overline{e-b} \times \overline{e-c} \times \overline{e-d}$ cannot be equal to nothing, but muſt be ſome real product: and therefore there is no ſuppoſition beſide one of the foreſaid four, that gives a juſt value of x according to the propoſed equation. So that it can have no more than theſe four roots. And after the ſame manner it appears, that "*No equation can have more roots than it contains dimenſions of the unknown quantity.*"

§ 6. To make all this ſtill plainer by an example, in numbers; ſuppoſe the equation to be reſolved to be $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$, and that you diſcover that this equation is the ſame with the product of $\overline{x-1} \times \overline{x-2} \times \overline{x-3} \times \overline{x-4}$, then you certainly infer that the four values of x are 1, 2, 3, 4; ſeeing any of theſe numbers placed for x makes that product, and conſequently $x^4 - 10x^3 + 35x^2 - 50x + 24$, equal to nothing, according to the propoſed equation. And it is certain that there can be no other values of x beſides theſe four: ſince when you ſubſtitute any other number for x in thoſe factors $x-1$, $x-2$, $x-3$, $x-4$, none of the factors vaniſh, and therefore their product cannot be equal to nothing according to the equation.

§ 7. It may be uſeful ſometimes to conſider equations as generated from others of an inferior ſort beſides ſimple ones. Thus a *cubic* equation may be conceived as generated from the *quadratic* $x^2 - px + q = 0$, and the *ſimple* equation $x - a = 0$, multiplied by each other; whoſe product

$$\left.\begin{matrix} x^3 - px^2 + qx - aq \\ \quad - ax^2 + apx \end{matrix}\right\} = 0$$ may expreſs any cubic equation whoſe roots are the quantity (a) the value of x in the ſimple equation, and the two roots of the quadratic equation, *viz.* $\frac{p + \sqrt{p^2 - 4q}}{2}$ and $\frac{p - \sqrt{p^2 - 4q}}{2}$; as appears from *Chap.* 13. *Part* I. And, according as theſe roots are *real* or *impoſſible*, two of the roots of the cubic equation are *real* or *impoſſible*.

§ 8. In the doctrine of involution we ſhewed that "the ſquare of any quantity poſitive or negative, is always poſitive," and therefore "the ſquare root of a negative is impoſſible or imaginary." For example, the $\sqrt{a^2}$ is either $+a$ or $-a$; but $\sqrt{-a^2}$ can neither be $+a$ nor $-a$, but muſt be *imaginary*. Hence is underſtood that "a quadratic equation may have no impoſſible expreſſion in its coefficients, and yet, when it is reſolved into the ſimple equations that produce it, they may involve impoſſible expreſſions." Thus the quadratic equation $x^2 + a^2 = 0$ has no impoſſible coefficient, but the ſimple equations from which it is produced, *viz.* $x + \sqrt{-a^2} = 0$, and $x - \sqrt{-a^2} = 0$, both involve an imaginary quantity; as the ſquare $-a^2$ is a real quantity, but its ſquare root is imaginary. After the ſame manner a biquadratic equation, when reſolved, may give four ſimple equations, each of which may give an impoſſible value for the root: and the ſame may be ſaid of any equation that can be produced from quadratic equations only; that is, whoſe dimenſions are of the even numbers.

§ 9. But "a cubic equation (which cannot be generated from quadratic equations only, but requires one ſimple equation beſides to produce it) if none of its coefficients are impoſſible, will have, at leaſt, one real root," the ſame with the root of the ſimple equation whence it is produced. The ſquare of an impoſſible quantity may be real, as the ſquare of $\sqrt{-a^2}$ is $-a^2$; but "the cube of an impoſſible quantity is ſtill impoſſible," as it ſtill involves the ſquare root of a negative: as, $\sqrt[2]{-a^2} \times \sqrt[2]{-a^2} \times \sqrt[2]{-a^2} = \sqrt[2]{-a^6} = a^3\sqrt{-1}$, is plainly imaginary. From which it appears, that though two ſimple equations involving impoſſible expreſſions, multiplied by one another, may give a product where no impoſſible expreſſion may appear; yet "if three ſuch ſimple equations be multiplied by each other, the impoſſible expreſſion will not diſappear in their product." And hence it is plain, that though a quadratic equation whoſe coefficients are all real may have its two roots impoſſible, yet "a cubic equation whoſe coefficients are real cannot have all its three roots impoſſible."

§ 10. In general, it appears that the impoſſible expreſſions cannot diſappear in the equation produced, but when their number is *even*; that there are never in any equations, whoſe coefficients are real quantities, ſingle impoſſible roots, or an odd number of impoſſible roots, but "that the roots become impoſſible in pairs;" and that "an equation of an odd number of dimenſions has always one real root."

§ 11. The roots of equations are either *poſitive* or *negative* according as the roots of the ſimple equations whence they are produced are poſitive or negative." If you ſuppoſe $x=-a$, $x=-b$, $x=-c$, $x=-d$, &c. then ſhall $x+a=0$, $x+b=0$, $x+c=0$, $x+d=0$; and the equation $\overline{x+a}\times\overline{x+b}\times\overline{x+c}\times\overline{x+d}=0$ will have its roots, $-a$, $-b$, $-c$, $-d$, &c. negative.

But to know when the roots of equations are poſitive and when negative, and how many there are of each kind, ſhall be explained in the next chapter.

CHAP.

CHAP. II.

Of the Signs and Coefficients of EQUATIONS.

§ 12. WHEN any number of ſimple equations are multiplied by each other, it is obvious that the higheſt dimenſion of the unknown quantity in their product is equal to the number of thoſe ſimple equations; and, the term involving the higheſt dimenſion is called the *firſt* term of the equation generated by this multiplication. The term involving the next dimenſion of the unknown quantity, leſs than the greateſt by unit, is called the *ſecond* term of the equation; the term involving the next dimenſion of the unknown quantity, which is leſs than the greateſt by two, the *third* term of the equation, *&c.* And that term which involves no dimenſion of the unknown quantity, but is ſome known quantity, is called the *laſt* term of the equation.

" *The number of terms is always greater than the higheſt dimenſion of the unknown quantity by unit.*" And when any term is wanting, an *aſteriſk* is marked in its place. The *ſigns* and *coefficients* of equations will be underſtood by conſidering the following TABLE, where the ſimple equation $x - a$, $x - b$, &c. are multiplied by one another, and produce ſucceſſively the higher equations.

$$\begin{array}{l}
\overline{x-a}=0 \\
\times\ \overline{x-b}=0 \\
\hline
=\left.\begin{array}{l} x^2-ax \\ \quad -bx+ab \end{array}\right\}=0, \text{ a Quadratic.} \\
\times\ \overline{x-c}=0 \\
\hline
=x^3\left.\begin{array}{l} -a \\ -b \\ -c \end{array}\right\}\times x^2\left.\begin{array}{l} +ab \\ +ac \\ +bc \end{array}\right\}\times x-abc=0, \text{ a Cubic.} \\
\times\ \overline{x-d}=0 \\
\hline
=x^4\left.\begin{array}{l} -a \\ -b \\ -c \\ -d \end{array}\right\}\times x^3\left.\begin{array}{l} +ab \\ +ac \\ +ad \\ +bc \\ +bd \\ +cd \end{array}\right\}\times x^2\left.\begin{array}{l} -abc \\ -abd \\ -acd \\ -bcd \end{array}\right\}\times x+abcd=0, \text{ a Biquadratic.} \\
\times\ \overline{x-e}=0 \\
\hline
=x^5\left.\begin{array}{l} -a \\ -b \\ -c \\ -d \\ -e \end{array}\right\}\times x^4\left.\begin{array}{l} +ab \\ +ac \\ +ad \\ +ae \\ +bc \\ +bd \\ +be \\ +cd \\ +ce \\ +de \end{array}\right\}\times x^2\left.\begin{array}{l} -abc \\ -abd \\ -abe \\ -acd \\ -ade \\ -ace \\ -bcd \\ -bce \\ -bde \\ -cde \end{array}\right\}\times x^2\left.\begin{array}{l} +abcd \\ +abce \\ +abde \\ +acde \\ +bcde \end{array}\right\}\times x-abcde=0, \text{ a Surſolid.} \\
\qquad\qquad\qquad\qquad\qquad \text{\&c.}
\end{array}$$

§ 13.

§ 13. From the inſpection of theſe equations it is plain, that the coefficients of the firſt term is *unit.*

The coefficient of the ſecond term is *the ſum of all the roots (a, b, c, d, e) having their ſigns changed.*

The coefficient of the third term is *the ſum of all the products that can be made by multiplying any two of the roots (a, b, c, d, e) by one another.*

The coefficient of the fourth term is *the ſum of all the products that can be made by multiplying into one another any three of the roots, with their ſigns changed.* And after the ſame manner all the other coefficients are formed.

The laſt term is always *the product of all the roots having their ſigns changed,* multiplied by one another.

§ 14. Although in the Table ſuch ſimple equations only are multiplied by one another as have poſitive roots, it is eaſy to ſee, that "the coefficients will be formed according to the ſame rule when any of the ſimple equations have negative roots." And, *in general,* if $x^3 - px^2 + qx - r = 0$ repreſent any cubic equation, then ſhall p be the ſum of the roots; q the ſum of the products made by multiplying any two of them; r the product of all the three: and, if $-p$, $+q$, $-r$, $+s$, $-t$, $+u$, &c. be the coefficients of the 2d, 3d, 4th, 5th, 6th, 7th, *&c.*

&c. terms of any equation, then ſhall p be the ſum of all the roots, q the ſum of the products of any two, r the ſum of the products of any three, s the ſum of the products of any four, t the ſum of the products of any five, u the ſum of the products of any ſix, *&c.*

§ 15. When therefore any equation is propoſed to be reſolved, it is eaſy to find the ſum of the roots, (for it is equal to the coefficient of the ſecond term having its ſign changed :) or, to find the ſum of the products that can be made by multiplying any determinate number of them.

But it is alſo eaſy "to find the ſum of the ſquares, or of any powers, of the roots."

The ſum of the ſquares is always $p^2 - 2q$. For calling the ſum of the ſquares B, ſince the ſum of the roots is p; and "the ſquare of the ſum of any quantities is always equal to the ſum of their ſquares added to double the products that can be made by multiplying any two of them," therefore $p^2 = B + 2q$, and conſequently $B = p^2 - 2q$. For example, $\overline{a + b + c}|^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$; that is, $p^2 = B + 2q$. And $\overline{a + b + c + d}|^2 = a^2 + b^2 + c^2 + d^2 + 2 \times \overline{ab + ac + ad + bc + bd + cd}$, that is again, $p^2 = B + 2q$, or $B = p^2 - 2q$: And ſo for any other number of quantities. In general therefore, "B the ſum of the ſquares of the roots may always be found by ſubtracting $2q$ from p^2;" the quantities p and q being always known, ſince they are the coefficients in the propoſed equation.

§ 16. "The ſum of the cubes of the roots of any equation is equal to $p^3 - 3pq + 3r$, or to $Bp - pq + 3r$." For $\overline{B-q} \times p$ gives always the exceſs of the ſum of the cubes of any quantities above the triple ſum of the products that can be made by multiplying any three of them. Thus $\overline{a^2+b^2+c^2-ab-ac-bc} \times \overline{a+b+c}$ (= $\overline{B-q} \times p$) $= a^3 + b^3 + c^3 - 3abc$. Therefore if the ſum of the cubes is called C, then ſhall $\overline{B-q} \times p = C - 3r$, and $C = Bp - qp + 3r$ (becauſe $B = p^2 - 2q$) $= p^3 - 3pq + 3r$.

After the ſame manner, if D be the ſum of the 4th powers of the roots, you will find that $D = pC - qB + pr - 4s$: and if E be the ſum of the 5th powers then ſhall $E = pD - qC + rB - ps + 5t$. And after the ſame manner the ſum of any powers of the roots may be found; the progreſſion of theſe expreſſions of the ſum of the powers being obvious.

§ 17. As for the ſigns of the terms of the equation produced, it appears from inſpection that the ſigns of all the terms in any equation in the Table are alternately + and −: theſe equations are generated by multiplying continually $x - a$, $x - b$, $x - c$, $x - d$, &c. by one another. The firſt term is always ſome pure power of x, and is poſitive; the ſecond is a power of x multiplied by the quantities $-a$, $-b$, $-c$, &c. And ſince theſe are all negative, that term muſt therefore be negative. The third term has the products of any two of theſe quantities ($-a$, $-b$, $-c$, &c.) for its coefficient; which products are all poſitive, becauſe $- \times -$ gives $+$. For the like reaſon, the next coefficient, conſiſting of all the products made by multiplying any three of theſe quantities, muſt be negative; and the next poſitive. So that the coefficients. in this caſe, will be poſitive and negative by turns. But "in this caſe the roots are all poſitive;" ſince $x = a$, $x = b$, $x = c$, $x = d$, $x = e$, &c. are the aſſumed ſimple equations. It is plain then, that "*when all the roots are poſitive, the ſigns are alternately + and −.*"

§ 18. But if the roots are all negative, then $\overline{x+a} \times \overline{x+b} \times \overline{x+c} \times \overline{x+d}$, &c. $= 0$, will expreſs the equation to be produced; all whoſe terms will plainly be poſitive; ſo that "*when all the roots of an equation are negative, it is plain there will be no changes in the ſigns of the terms of that equation.*"

§ 19. In general, "there are as many poſitive roots in any equation as there are changes in the ſigns of the terms from + to −, or from − to +; and the remaining roots are negative."

negative." The Rule is general, if the impoſſible roots be allowed to be either poſitive or negative.

§ 20. In quadratic equations, the two roots are either both poſitive, as in this

$$(\overline{x-a}\times\overline{x-b}=)\,x^2\begin{array}{l}-ax\\-bx\end{array}\!\!\begin{array}{l}+ab=0,\\ \end{array}$$

where there are two changes of the ſigns: Or they are both negative, as in this

$$(\overline{x+a}\times\overline{x+b}=)\,x^2\left.\begin{array}{l}+a\\+b\end{array}\right\}x+ab=0.$$

where there is not any change of the ſigns. Or there is one poſitive and one negative, as in

$$(\overline{x-a}\times\overline{x+b}=)\,x^2\left.\begin{array}{l}-a\\+b\end{array}\right\}x-ab=0,$$

where there is neceſſarily one change of the ſigns; becauſe the firſt term is poſitive, and the laſt negative, and there can be but one change whether the ſecond term be + or −.

Therefore the rule given in the 19th ſection extends to all quadratic equations.

§ 21. In cubic equations, the roots may be,

1°. All poſitive as in this, $\overline{x-a}\times\overline{x-b}\times\overline{x-c}=0$, in which the ſigns are alternately + and −, as appears from the Table; and there are three changes of the ſigns.

2°. The roots may be all negative, as in the equation $\overline{x+a}\times\overline{x+b}\times\overline{x+c}=0$, where there can be no change of the ſigns. Or, 3°. There may be two poſitive roots and one negative, as in the equation $\overline{x-a}\times\overline{x-b}\times\overline{x+c}=0$, which gives

$$\left.\begin{array}{r}x^3-a\\-b\\+c\end{array}\right\}\left.\begin{array}{r}+ab\\x^2-ac\\-bc\end{array}\right\}x+abc=0.$$

Here there muſt be two changes of the ſigns: becauſe if $a+b$ is greater than c, the ſecond term muſt be negative, its coefficient being $-a$, $-b+c$.

And if $a+b$ is leſs than c, then the third term muſt be negative, its coefficient $+ab-ac-bc$ $(ab-c\times\overline{a+b})$ * being in that caſe negative. And there cannot poſſibly be three changes of the ſigns, the firſt and laſt terms having the ſame ſign.

4°. There may be one poſitive root and two negative, as in the equation $\overline{x+a}\times\overline{x+b}\times\overline{x-c}=0$, which gives

$$\left.\begin{array}{r}x^3+a\\+b\\-c\end{array}\right\}\left.\begin{array}{r}+ab\\x^2-ac\\-bc\end{array}\right\}x-abc=0.$$

Where there muſt be always one change of the ſigns, ſince the firſt term is poſitive and the laſt negative. And there can be but one change of the ſigns, ſince if the ſecond term is negative, or $a+b$ leſs than c, the third muſt be

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*Becauſe the rectangle $a\times b$ is leſs than the ſquare $\overline{a+b}\times\overline{a+b}$, and therefore much leſs than $\overline{a+b}\times c$.

negative alſo, ſo that there will be but one change of the ſigns. Or, if the ſecond term is affirmative, whatever the third term is, there will be but one change of the ſigns. It appears therefore, in general, that in cubic equations; there are as many affirmative roots as there are changes of the ſigns of the terms of the equation.

The ſame way of reaſoning may be extended to equations of higher dimenſions, and the rule delivered in § 19, extended to all kinds of equations.

§ 22. There are ſeveral conſectaries of what has been already demonſtrated, that are of uſe in diſcovering the roots of equations. But before we proceed to that, it will be convenient to explain ſome transformations of equations, by which they may often be rendered more ſimple, and the inveſtigation of their roots more eaſy.

CHAP. III.

Of the Transformation of EQUATIONS; and exterminating their intermediate terms.

§ 23. WE now proceed to explain the transformations of equations that are moſt uſeful: and firſt, "*The affirmative roots of an equation are changed into negative roots of the ſame value, and the negative roots into affirmative, by only changing the ſigns of the terms alternately, beginning with the ſecond.*" Thus the roots of the equation $x^4 - x^3 - 19x^2 + 49x - 30 = 0$ are $+1, +2, +3, -5$; whereas the roots of the ſame equation having only the ſigns of the ſecond and fourth terms changed, *viz.* $x^4 + x^3 - 19x^2 - 49x - 30 = 0$, are $-1, -2, -3, +5$.

To underſtand the reaſon of this rule, let us aſſume an equation, as $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d} \times \overline{x-e}$, &c. $= 0$, whoſe roots are $+a$, $+b$, $+c$, $+d$, $+e$, &c. and another having its roots of the ſame value, but affected with contrary ſigns, as $\overline{x+a} \times \overline{x+b} \times \overline{x+c} \times \overline{x+d} \times \overline{x+e}$, &c. $= 0$. It is plain, that the terms taken alternately, beginning from the firſt, are the ſame in both equations, and have the ſame ſign, "being products of an even number of the roots;" the product of any two roots having the ſame ſign as their product when both their ſigns are changed; as $+a \times -b = -a \times +b$.

But the ſecond terms and all taken alternately from them, becauſe their coefficients involve always the products of an odd number of the roots, will have contrary ſigns in the two equations. For example, the product of four, *viz.* $abcd$, having the ſame ſign in both, and one equation in the fifth term having $abcd \times +e$, and the other $abcd \times -e$, it follows that their product $abcde$ muſt have contrary ſigns in the two equations: theſe two equations therefore that have the ſame roots, but with contrary ſigns, have nothing different but the ſigns of the alternate terms, beginning with the ſecond. From which it follows, "that if any equation is given, and you change the ſigns of the alternate terms, beginning with the ſecond, the new equation will have roots of the ſame value, but with contrary ſigns."

§ 24. It is often very uſeful "*to transform an equation into another that ſhall have its roots greater or leſs than the roots of the propoſed equation by ſome given difference.*"

Let the equation propoſed be the cubic $x^3 - px^2 + qx - r = 0$. And let it be required to transform it into another equation whoſe roots ſhall be leſs than the roots of this equation by ſome given difference (e), that is, ſuppoſe $y = x - e$, and conſequently $x = y + e$; then inſtead of x and its powers, ſubſtitute $y + e$ and its powers, and there will ariſe this new equation.

$$(A)\left.\begin{array}{rrrr} y^3 + 3ey^2 & + 3e^2y & + e^3 \\ - py^2 & - 2pey & - pe^2 \\ & + qy & + qe \\ & & - r \end{array}\right\} = 0.$$

whoſe roots are leſs than the roots of the preceding equation by the difference (e).

If it had been required to find an equation whoſe roots ſhould be greater than thoſe of the propoſed equation by the quantity (e), then we muſt have ſuppoſed $y = x + e$, and conſequently $x = y - e$, and then the other equation would have had this form;

$$(B)\left.\begin{array}{rrrr} y^3 - 3ey^2 & + 3e^2y & - e^3 \\ - py^2 & + 2pey & - pe^2 \\ & + qy & - qe \\ & & - r \end{array}\right\} = 0.$$

If the propoſed equation be in this form, $x^3 + px^2 + qx + r = 0$, then by ſuppoſing $x + e = y$ there will ariſe an equation agreeing in all reſpects with the equation (A), but that the ſecond and fourth terms will have contrary ſigns.

And by ſuppoſing $x - e = y$, there will ariſe an equation agreeing with *(B)* in all reſpects, but that the ſecond and fourth terms will have contrary ſigns to what they have in *(B)*.

The firſt of theſe ſuppoſitions gives this equation,

$$(C)\ \left.\begin{array}{rrrr} y^3 & - 3ey^2 & + 3e^2y & - e^3 \\ & + py^2 & - 2pey & + pe^2 \\ & & + qy & - qe \\ & & & + r \end{array}\right\} = 0.$$

The ſecond ſuppoſition gives the equation

$$(D)\ \left.\begin{array}{rrrr} y^3 & + 3ey^2 & + 3e^2y & + e^3 \\ & + py^2 & + 2epy & + pe^2 \\ & & + qy & + qe \\ & & & + r \end{array}\right\} = 0,$$

§ 25. The firſt uſe of this transformation of equations is to ſhew "*how the ſecond (or other intermediate) term may be taken away out of an equation.*"

It is plain that in the equation *(A)* whoſe ſecond term is $\overline{3e-p} \times y^2$, if you ſuppoſe $e = \frac{1}{3}p$, and conſequently $3e - p = 0$, then the ſecond term will vaniſh.

In the equation *(C)* whoſe ſecond term is $-\overline{3e+p} \times y^2$, ſuppoſing $e = \frac{1}{3}p$, the ſecond term alſo vaniſhes.

Now the equation *(A)* was deduced from $x^3 - px^2 + qx - r = 0$, by ſuppoſing $y = x - e$:

and the equation *(C)* was deduced from $x^3+px^2+qx+r=0$; by ſuppoſing $y=x+e$. From which this Rule may eaſily be deduced for exterminating the ſecond term out of any cubic equation.

RULE.

"*Add to the unknown quantity of the given equation the third part of the coefficient of the ſecond term with its proper ſign,* viz. $\mp\frac{1}{3}p$, *and ſuppoſe this aggregate equal to a new unknown quantity* (*y*). *From this value of y find a value of x by tranſpoſition, and ſubſtitute this value of x and its powers in the given equation, and there will ariſe a new equation that ſhall want the ſecond term.*"

EXAMPLE.

Let it be required to exterminate the ſecond term out of this equation, $x^3-9x^2+26x-34=0$, ſuppoſe $x-3=y$, or $y+3=x$; and ſubſtituting according to the Rule, you will find

$$\left.\begin{array}{rrrr} y^3 & +9y^2 & +27y & +27 \\ & -9y^2 & -54y & -81 \\ & & +26y & +78 \\ & & & -34 \end{array}\right\}=0$$

$$y^3 \quad * \quad -y \quad -10=0.$$

In which there is no term where y is of two dimenſions, and an aſteriſk is placed in the room of the ſecond term, to ſhew it is wanting.

§ 26. Let the equation propoſed be of any number of dimenſions repreſented by (n); and let the coefficient of the ſecond term with its ſign prefixed be $-p$, then ſuppoſing $x-\frac{p}{n}=y$, and conſequently $x=y+\frac{p}{n}$, and ſubſtituting this value for x in the given equation, there will ariſe a new equation that ſhall want the ſecond term.

It is plain from what was demonſtrated in *Chap.* 2. that the ſum of the roots of the propoſed equation is $+p$; and ſince we ſuppoſe $y=x-\frac{p}{n}$, it follows, that in the new equation, each value of y will be leſs than the reſpective value of x by $\frac{p}{n}$; and, ſince the number of the roots is n, it follows that the ſum of the values of y will be leſs than $+p$, the ſum of the values of x, by $n\times\frac{p}{n}$ the difference of any two roots, that is, by $+p$: therefore the ſum of the values of y will be $+p-p=0$.

But the coefficient of the ſecond term of the equation of y is the ſum of the values of y, viz. $+p-p$, and therefore that coefficient is equal to nothing; and conſequently, in the equation of y, the ſecond term vaniſhes. It follows then, that the ſecond term may be exterminated out of any given equation by the following

RULE.

"*Divide the coefficient of the ſecond term of the propoſed equation by the number of dimenſions of the equation; and aſſuming a new unknown quantity y, add to it the quotient having its ſign changed. Then ſuppoſe this aggregate equal to x the unknown quantity in the propoſed equation; and for x and its powers, ſubſtitute the aggregate and its powers, ſo ſhall the new equation that ariſes want its ſecond term.*"

§ 27. If the propoſed equation is a quadratic, as $x^2 - px + q = 0$, then, according to the rule, ſuppoſe $y + \frac{1}{2}p = x$, and ſubſtituting this value for x, you will find,

$$\left.\begin{array}{rrr} y^2 & + py & + \frac{1}{4}p^2 \\ & - py & - \frac{1}{2}p^2 \\ & & + q \end{array}\right\} = 0$$

$$\overline{y^2 \quad * \quad - \tfrac{1}{4}p^2 + q = 0.}$$

And from this example the uſe of exterminating the ſecond term appears: for commonly the ſolution of the equation that wants the ſecond term is more eaſy. And, if you can find the value of y from this new equation, it is eaſy to find the value of x by means of the equation $y+\frac{1}{2}p=x$. For example,

Since $y^2+q-\frac{1}{4}p^2=0$, it follows that

$y^2=\frac{1}{4}p^2-q$, and $y=\pm\sqrt{\frac{1}{4}p^2-q}$, ſo that $x=y+\frac{1}{2}p=\frac{1}{2}p\pm\sqrt{\frac{1}{4}p^2-q}$;

which agrees with what we demonſtrated, *Chapter* 13. *Part.* I.

If the propoſed equation is a biquadratic, as $x^4-px^3+qx^2-rx+s=0$, then by ſuppoſing $x-\frac{1}{4}p=y$, or $x=y+\frac{1}{4}p$, an equation ſhall ariſe having no ſecond term. And if the propoſed is of five dimenſions, then you muſt ſuppoſe $x=y\pm\frac{1}{5}p$. And ſo on.

§ 28. When the ſecond term in any equation is wanting, it follows, that, "the equation has both affirmative and negative roots," and that the ſum of the affirmative roots is equal to the ſum of the negative roots: by which means the coefficient of the ſecond term, which is the ſum of all the roots of both ſorts, vaniſhes, and makes the ſecond term vaniſh.

In general, "the coefficient of the ſecond term is the difference between the ſum of the affirmative roots and the ſum of the negative roots:" and the operations we have given ſerve only to diminiſh all the roots when the ſum of the affirmative is greateſt, or increaſe the roots when the ſum of the negative is greateſt, ſo as to balance them, and reduce them to an equality.

It is obvious, that in a quadratic equation that wants the ſecond term, there muſt be one root affirmative and one negative; and theſe muſt be equal to one another.

In a cubic equation that wants the ſecond term, there muſt be either, two affirmative roots equal, taken together, to a third root that muſt be negative; or, two negative equal to a third that muſt be poſitive.

"Let an equation $x^3 - px^2 + qx - r = 0$ be propoſed, and let it be now required to exterminate the third term."

By ſuppoſing $y = x - e$, the coefficient of the third term in the equation of y is found (ſee equation *A*) to be $3e^2 - 2pe + q$. Suppoſe that coefficient equal to nothing, and by reſolving the quadratic equation $3e^2 - 2pe + q = 0$, you will find the value of e, which ſubſtituted for it in the equation $y = x - e$, will ſhew how to tranſform the propoſed equation into one that ſhall want the third term.

The quadratic $3e^2 - 2pe + q = 0$ gives $e = \frac{p \pm \sqrt{p^2 - 3q}}{3}$. So that the propoſed cubic will be transformed into an equation wanting the third term by ſuppoſing $y = x - \frac{p - \sqrt{p^2 - 3q}}{3}$, or $y = x - \frac{p + \sqrt{p^2 - 3q}}{3}$.

If the propoſed equation is of n dimenſions, the value of e, by which the third term may be taken away, is had by reſolving the quadratic equation $e^2 + \frac{2p}{n} \times e + \frac{2q}{n \times \overline{n-1}} = 0$, ſuppoſing $-p$ and $+q$ to be the coefficients of the ſecond and third terms of the propoſed equation.

The fourth term of any equation may be taken away by ſolving a cubic equation, which is the coefficient of the fourth term in the equation when transformed, as in the ſecond article of this chapter. The fifth term may be taken away by ſolving a biquadratic; and after the ſame manner the other terms can be exterminated if there are any.

§ 29. There are other tranſmutations of equations, that on ſome occaſions are uſeful.

An equation, as $x^3 - px^2 + qx - r = 0$, *may be tranſformed into another that ſhall have its roots equal to the roots of this equation multiplied by a given quantity*, as f, by ſuppoſing $y = fx$, and conſequently $x = \frac{y}{f}$, and ſubſtituting this value for x in the propoſed equation, there will ariſe $\frac{y^3}{f^3} - \frac{py^2}{f^2} + \frac{qy}{f} - r = 0$, and multiplying all by $f^3 \ldots y^3 - fpy^2 + f^2qy - f^3r = 0$, where the coefficient of the ſecond term of the propoſed equation multiplied into f makes the coefficient of the ſecond term of the transformed equation; and the following coefficients are produced by the following coefficients of the propoſed equation (as q, r, &c.) multiplied into the powers of f (f^2, f^3, &c.)

Therefore "to transform any equation into another whoſe roots ſhall be equal to the roots of the propoſed equation multiplied by a given quantity" (f), you need only multiply the terms of the propoſed equation, beginning at the ſecond term, by f, f^2, f^3, f^4, &c. and putting y inſtead of x there will ariſe an equation having its roots equal to the roots of the propoſed equation multiplied by (f) as required.

§ 30. The transformation mentioned in the laſt article is of uſe when the higheſt term of the equation has a coefficient different from unity; for, by it, the equation may be transformed into one that ſhall have the coefficient of the higheſt term unit.

If the equation propoſed is $ax^3 - px^2 + qx - r = 0$, then transform the equation into one whoſe roots are equal to the roots of the propoſed equation multiplied by (a). That is, ſuppoſe $y = ax$, or $x = \frac{y}{a}$, and there will ariſe $\frac{ay^3}{a^3} - \frac{py^2}{a^2} + \frac{qy}{a} - r = 0$; ſo that $y^3 - py^2 + qay - ra^2 = 0$.

From which we eaſily draw this

RULE.

"*Change the unknown quantity* x *into another* y, *prefix no coefficient to the higheſt term, paſs the ſecond, multiply the following terms, beginning with the third, by* a, a^2, a^3, a^4, &c. *the powers of the coefficient of the higheſt term of the propoſed equation, reſpectively.*"

Thus the equation $3x^3 - 13x^2 + 14x + 16 = 0$, is transformed into the equation
$y^3 - 13y^2 + 14 \times 3 \times x + 16 \times 9 = 0$, or
$y^3 - 13y^2 + 42x + 144 = 0$.

Then finding the roots of this equation, it will eaſily be diſcovered what are the roots of the propoſed equation: ſince $3x = y$, or $x = \frac{1}{3}y$. And therefore ſince one of the values of y is -2, it follows that one of the values of x is $-\frac{2}{3}$.

§ 31. By the laſt Rule "*an equation is eaſily cleared of fractions.*" Suppoſe the equation propoſed is $x^3 - \frac{p}{m}x^2 + \frac{q}{n}x - \frac{r}{e} = 0$. Multiply all the terms by the product of the denominators, you find

$$mne \times x^3 - nep \times x^2 + meq \times x - mnr = 0:$$

Then (by laſt ſection) transforming the equation into one that ſhall have unit for the coefficient of the higheſt term, you find

$$y^3 - nep \times y^2 + m^2e^2nq \times y - m^3n^3e^2r = 0.$$

Or, neglecting the denominator of the laſt term $\frac{r}{e}$, you need only multiply all the equation by *mn*, which will give $mn \times x^3 - np \times x^2 + mq \times x - \frac{mnr}{e} = 0$. And then $y^3 - np \times y^2 + m^2nq \times y - \frac{m^3n^3r}{e} = 0$.

Now after the values of y are found, it will be eaſy to diſcover the values of x; ſince, in the firſt caſe, $x = \frac{y}{mne}$; in the ſecond, $x = \frac{y}{mn}$.

For example, the equation

$x^3 * - \frac{4}{3}x - \frac{146}{27} = 0$, is firſt reduced to this form $3x^3 * - 4x - \frac{146}{9} = 0$, and then transformed into $y^3 * - 12y - 146 = 0$.

Sometimes, by theſe transformations, "*Surds are taken away.*" As for example,

The equation $x^3 - p\sqrt{a} \times x^2 + qx - r\sqrt{a} = 0$, by putting $y = \sqrt{a} \times x$, or $x = \frac{y}{\sqrt{a}}$, is transformed into this equation,

$$\frac{y^3}{a\sqrt{a}} - p\sqrt{a} \times \frac{y^2}{a} + q \times \frac{y}{\sqrt{a}} - r\sqrt{a} = 0.$$

Which by multiplying all the terms by $a\sqrt{a}$, becomes $y^3 - pay^2 + qay - ra^2 = 0$, an equation free of ſurds. But in order to make this ſucceed, the ſurd ($\sqrt{a}$) muſt enter the alternate terms beginning with the ſecond.

§ 32. An equation, as $x^3-px^2+qx-r=0$, *may be transformed into one whoſe roots ſhall be the quantities reciprocal of* x; by ſuppoſing $y=\frac{1}{x}$, and $y=\frac{z}{r}$, or, (by one ſuppoſition) $x=\frac{r}{z}$, becomes $z^3-qz^2+prz-r^2=0$.

In the equation of y, it is manifeſt that the order of the coefficients is inverted; ſo that if the ſecond term had been wanting in the propoſed equation, the laſt but one ſhould have been wanting in the equations of y and z. If the third had been wanting in the equation propoſed, the laſt but two had been wanting in the equations of y and z.

Another uſe of this transformation is, that "*the greateſt root in the one is transformed into the leaſt root in the other.*" For ſince $x=\frac{1}{y}$, and $y=\frac{1}{x}$, it is plain that when the value of x is greateſt, the value of y is leaſt, and converſely.

How an equation is transformed ſo as to have all its roots affirmative, ſhall be explained in the following chapter.

CHAP.

CHAP. IV.

Of finding the Roots of Equations when two or more of the Roots are equal to each other.

§ 33. BEFORE we proceed to explain how to reſolve equations of all ſorts, we ſhall firſt demonſtrate "*how an equation that has two or more roots equal, is depreſſed to a lower dimenſion*;" and its reſolution made, conſequently, more eaſy. And ſhall endeavour to explain the grounds of this and many other rules we ſhall give in the remaining part of this Treatiſe, in a more ſimple and conciſe manner than has hitherto been done.

In order to this, we muſt look back to § 24. where we find that if any equation, as $x^3 - px^2 + qx - r = 0$, is propoſed, and you are to tranſform it into another that ſhall have its roots leſs than the values of x by any given difference, as e, you are to aſſume $y = x - e$, and ſubſtituting for x its value $y + e$, you find the tranſformed equation,

$$\left.\begin{array}{rrrr} y^3 + 3ey^2 & + 3e^2y & + e^3 \\ - py^2 & - 2pey & - pe^2 \\ & + qy & + qe \\ & & - r \end{array}\right\} = 0.$$

Where we are to obſerve,

1° That the laſt term $(e^3 - pe^2 + qe - r)$ is the very equation that was propoſed, having e in place of x.

2° The coefficient of the laſt term but one is $3e^2 - 2pe + q$, which is the quantity that ariſes by multiplying every term of the laſt coefficient $e^3 - pe^2 + qe - r$ by the index of e in each term, and dividing the product $3e^3 - 2pe^2 + qe$ by the quantity e that is common to all the terms.

3° The coefficient of the laſt term but two is $3e - p$, which is the quantity that ariſes by multiplying every term of the cofficient laſt found $(3e^2 - 2pe + q)$ by the index of e in each term, and dividing the whole by $2e$.

§ 34. Theſe ſame obſervations extend to equations of all dimenſions. If it is the biquadratic $x^4 - px^3 + qx^2 - rx + s = 0$ that is propoſed, then by ſuppoſing $y = x - e$, it will be transformed into this other,

$$\left.\begin{array}{rrrrr} y^4 & + 4ey^3 & + 6e^2y^2 & + 4e^3y & + e^4 \\ & - py^3 & - 3pey^2 & - 3pe^2y & - pe^3 \\ & & + qy^2 & + 2qey & + qe^2 \\ & & & - ry & - re \\ & & & & + s \end{array}\right\} = 0.$$

Where again it is obvious that the laſt term is the equation that was propoſed, having e in place of x. That the laſt term but one has for its coefficient the quantity that ariſes by multiplying the terms of the laſt quantity by the indices of e in each term, and dividing the product by e. That the coefficient of the laſt term but two (*viz.* $6e^2 - 3pe + q$) is deduced in the ſame manner from the term immediately following; that is, by multiplying every term of $4e^3 - 3pe^2 + 2qe - r$ by the index of e in that term, and dividing the whole by e multiplied into the index of y in the term ſought, that is, by $e \times 2$. And the next term is $4e - p = \frac{6e^2 \times 2 - 3pe \times 1}{3e}$.

The demonſtration of this may eaſily be made general by the Theorem for finding the powers of a binomial, ſince the transformed equation conſiſts of the powers of the binomial $y+e$ that are marked by the indices of e in the laſt term, multiplied each by their coefficients 1, $-p$, $+q$, $-r$, $+s$, &c. reſpectively.

§ 35. From the laſt two articles we can eaſily find the terms of the transformed equation without any evolution. The laſt term is had by ſubſtituting e inſtead of x in the propoſed equation; the next term, by multiplying every part of that laſt term by the index of e in each part, and dividing the whole by e; and the following terms in the manner deſcribed in the foregoing article; the reſpective diviſors being the quantity e multiplied by the index of y in each term.

The demonſtration for finding when two or more roots are equal will be eaſy, if we add to this, that "*when the unknown quantity enters all the terms of any equation, then one of its values is equal to nothing.*" As in the equation $x^3 - px^2 + qx = 0$, where $x - 0 = 0$ being one of the ſimple equations that produce $x^3 - px^2 + qx = 0$, it follows that one of the values of x is 0. In like manner two of the values of x are equal to nothing in this equation $x^3 - px^2 = 0$; and three of them vaniſh in the equation $x^4 - px^3 = 0$.

It is alſo obvious (*converſely*) that "if x does not enter all the terms of the equation, *i. e.* if the laſt term be not wanting, then none of the values of x can be equal to nothing;" for if every term be not multiplied by x, then $x - 0$ cannot be a diviſor of the whole equation, and conſequently 0 cannot be one of the values of x. If x^2 does not enter into all the terms of the equation, then two of the values of x, cannot be equal to nothing. If x^3 does not enter into all the terms of the equation, then three of the values of x cannot be equal to nothing, &c.

§ 36. Suppoſe now that two values of x are equal to one another, and to e; then it is plain that two values of y in the transformed equation will be equal to nothing: ſince $y = x - e$. And conſequently, by the laſt article, the two laſt terms of the transformed equation muſt vaniſh.

Suppoſe it is the cubic equation of § 33. that is propoſed, *viz.* $x^3 - px^2 + qx - r = 0$; and becauſe we ſuppoſe $x = e$, therefore the laſt term of the transformed equation, *viz* $e^3 - pe^2 + qe - r$ will vaniſh. And ſince two values of y vaniſh, the laſt term but one, *viz.* $3e^2y - 2pey + qy$ will vaniſh at the ſame time. So that $3e^2 - 2pe + q = 0$. But, by ſuppoſition, $e = x$; therefore, when two values of x, in the equation $x^3 - px^2 + qx - r = 0$, are equal, it follows, that $3x^2 - 2px + q = 0$. And thus "the propoſed cubic is depreſſed to a quadratic that has one of its roots equal to one of the roots of that cubic."

If it is the biquadratic that is propoſed, *viz.* $x^4 - px^3 + qx^2 - rx + s = 0$, and two of its roots be equal; then ſuppoſing $e = x$, two of the values of y muſt vaniſh, and the equation of § 34 will be reduced to this form,

$$\left.\begin{array}{r} y^4 + 4ey^3 + 6e^2y^2 \\ - py^3 - 3pey^2 \\ + qy^2 \end{array}\right\} * * = 0. \text{ So that}$$

$4e^3 - 3pe^2 + 2qe - r = 0$; or, becauſe $x = e$, $4x^3 - 3px^2 + 2qx - r = 0$.

In general, when two values of x are equal to each other, and to e, the two laſt terms of the transformed equation vaniſh: and conſequently, "if you multiply the terms of the propoſed equation by the indices of x in each term, the quantity that will ariſe will be $= 0$, and will give an equation of a lower dimenſion than the propoſed, that ſhall have one of its roots equal to one of the roots of the propoſed equation."

That the laſt two terms of the equation vaniſh when the values of x are ſuppoſed equal to each other, and to e, will alſo appear by conſidering, that ſince two values of y then become equal to nothing, the product of the values of y muſt vaniſh, which is equal to the laſt term of the equation; and becauſe two of the four values of y are equal to nothing, it follows alſo that one of any three that can be taken out of theſe four muſt be $= 0$; and therefore, the products made by multiplying any three muſt vaniſh; and conſequently the coefficient of the laſt term but one, which is equal to the ſum of theſe products, muſt vaniſh.

§ 37. After the ſame manner, if there are three equal roots in the biquadratic $x^4 - px^3 + qx^2 - rx + s = 0$, and if e be equal to one of them; three values of y $(= x - e)$ will vaniſh, and conſequently y^3 will enter all the terms of the transformed equation; which will have this form,

$$\left.\begin{array}{r}y^4+4ey^3\\-py^3\end{array}\right\}***=0.$$ So that here $6e^2-3pe+q=0$; or, ſince $e=x$, therefore, $6x^2-3px+q=0$: and one of the roots of this quadratic will be equal to one of the roots of the propoſed biquadratic.

In this caſe, two of the roots of the cubic equation $4x^3-3px^2+2qx-r=0$ are roots of the propoſed biquadratic, becauſe the quantity $6x^2-3px+q$ is deduced from $4x^3-3px^2+2qx-r$, by multiplying the terms by the indexes of x in each term.

In general, "whatever is the number of equal roots in the propoſed equation, they will all remain but one in the equation that is deduced from it, by multiplying all the terms by the indexes of x in them; and they will all remain but two in the equation deduced in the ſame manner from That;" and ſo of the reſt.

§ 38. What we obſerved of the coefficients of equations transformed by ſuppoſing $y=x-e$, leads to this eaſy demonſtration of this Rule; and will be applied in the next chapter to demonſtrate the Rules for finding the limits of equations.

It is obvious however, that though we make uſe of equations whoſe ſigns change alternately, the ſame reaſoning extends to all other equations.

It is a conſequence alſo of what has been demonſtrated, that “ if two roots of any equation, as $x^3 - px^2 + qx - r = 0$, are equal, then multiplying the terms by any arithmetical ſeries, as $a + 3b$, $a + 2b$, $a + b$, a, the product will be $= 0$.

For ſince $ax^3 - apx^2 + aqx - ar = 0$; and $\overline{3x^2 - 2px + q} \times bx = 0$, it follows that
$ax^3 + 3bx^3 - apx^2 - 2bpx^2 + aqx + bqx - ar = 0$.
Which is the product that ariſes by multiplying the terms of the propoſed equation by the terms of the ſeries, $a + 3b$, $a + 2b$, $a + b$, a; which may repreſent any arithmetical progreſſion.

CHAP. V.

Of the Limits of EQUATIONS.

§ 39. WE now proceed to ſhew how to diſcover the limits of the roots of equations, by which their ſolution is much facilitated.

Let any equation, as $x^3 - px^2 + qx - r = 0$, be propoſed; and transform it, as above, into the equation

$$\left.\begin{array}{rrrr} y^3 & + 3ey^2 & + 3e^2y & + e^3 \\ & - py^2 & - 2pey & - pe^2 \\ & & + qy & + qe \\ & & & - r \end{array}\right\} = 0.$$

Where the values of y are leſs than the reſpective values of x by the difference e. If you ſuppoſe e to be taken ſuch as to make all the coefficients, of the equation of y, poſitive, *viz.* $e^3 - pe^2 + qe - r$, $3e^2 - 2pe + q$, $3e - p$; then there being no variation of the ſigns in the equation, all the values of y muſt be negative; and conſequently, the quantity e, by which the values of x are diminiſhed, muſt be greater than the greateſt poſitive value of x: and conſequently muſt be the limit of the roots of the equation $x^3 - px^2 + qx - r = 0$.

It is ſufficient therefore, in order to find the limit, to "enquire what quantity ſubſtituted for x in each of theſe expreſſions $x^3 - px^2 + qx - r$, $3x^2 - 2px + q$, $3x - p$, will give them all poſitive;" for that quantity will be the limit required.

How theſe expreſſions are formed from one another, was explained in the beginning of the laſt chapter.

EXAMPLE.

§ 40. If the equation $x^5 - 2x^4 - 10x^3 + 30x^2 + 63x + 120 = 0$ is propoſed; and it is required to determine the limit that is greater than any of the roots; you are to enquire what integer number ſubſtituted for x in the propoſed equation, and following equations deduced from it by § 35, will give, in each, a poſitive quantity.

$$5x^4 - 8x^3 - 30x^2 + 60x + 63$$
$$5x^3 - 6x^2 - 15x + 15$$
$$5x^2 - 4x - 5$$
$$5x - 2.$$

The leaſt integer number which gives each of theſe poſitive, is 2; which therefore is the limit of the roots of the propoſed equation; or a number that exceeds the greateſt poſitive root.

If the limit of the *negative* roots is required, you may, by § 23, change the negative into poſitive roots, and then proceed as before to find their limits. Thus, in the example, you will find that -3 is the limit of the negative roots. So that the five roots of the propoſed equation are betwixt -3 and $+2$.

§ 41. Having found the limit that ſurpaſſes the greateſt poſitive root, call it m. And if you aſſume $y=m-x$, and for x ſubſtitute $m-y$, the equation that will ariſe will have all its roots poſitive; becauſe m is ſuppoſed to ſurpaſs all the values of x, and conſequently $m-x\ (=y)$ muſt always be affirmative. And by this means, *any equation may be changed into one that ſhall have all its roots affirmative.*

Or if $-n$ repreſent the limit of the negative roots, then by aſſuming $y=x+n$, the propoſed equation ſhall be transformed into one that ſhall have all its roots affirmative; for $+n$ being greater than any negative value of x, it follows, that $y=x+n$ muſt be always poſitive.

§ 42. "*The greateſt negative coefficent of any equation increaſed by unit, always exceeds the greateſt root of the equation.*"

To demonſtrate this, let the cubic $x^3-px^2-qx-r=0$ be propoſed; where all the terms are negative except the firſt. Aſſuming $y=x-e$ it will be transformed into the following equation.

$$(A)\left.\begin{array}{llll} y^3 + 3ey^2 & + 3e^2y & + e^3 \\ \quad - py^2 & - 2pey & - pe^2 \\ & - qy & - qe \\ & & - r \end{array}\right\} = 0,$$

1°. Let us ſuppoſe that the coefficients p, q, r, are equal to each other; and if you alſo ſuppoſe $e = p + 1$, then the laſt equation becomes

$$(B)\left.\begin{array}{llll} y^3 + 2py^2 & + p^2y & + 1 \\ \quad + 3y^2 & + 3py \\ & + 3y \end{array}\right\} = 0,$$

Where all the terms being poſitive, it follows that the values of y are all negative, and that conſequently e, or $p + 1$, is greater than the greateſt value of x in the propoſed equation.

$y = x - e \therefore x = e + (y)$ [illegible]

2°. If q and r be not $= p$, but leſs than it, and for e you ſtill ſubſtitute $p + 1$ (ſince the negative part $\left(\begin{array}{ll} - qy & - qe \\ & - r \end{array}\right)$ becomes leſs, the poſitive remaining undiminiſhed) *a fortiori*, all the coefficients of the equation (A) become poſitive. And the ſame is obvious if q and r have poſitive ſigns, and not negative ſigns, as we ſuppoſed. It appears therefore, “that, if, in any cubic equation, p be the greateſt negative coefficient, then $p + 1$ muſt ſurpaſs the greateſt value of x.”

§ 43. 3°. By the ſame reaſoning it appears, that if q be the greateſt negative coefficient of the equation, and $e = q+1$, then there will be no variation of the ſigns in the equation of y: for it appears from the laſt article, that if all the three (p, q, r) were equal to one another, and e equal to any one of them increaſed by unit, as to $q+1$, then all the terms of the equation (A) would be poſitive. Now if e be ſuppoſed ſtill equal to $q+1$, and p and r to be leſs than q, then, *a fortiori*, all theſe terms will be poſitive, the negative part, which involves p and r being diminiſhed, while the poſitive part and the negative involving q remain as before.

4°. After the ſame manner it is demonſtrated, that if r is the greateſt negative coefficient in the equation, and e is ſuppoſed $= r+1$, then all the terms of the equation (A) of y will be poſitive; and conſequently $r+1$ will be greater than any of the values of x.

What we have ſaid of the *cubic* equation $x^3 - px^2 + qx - r = 0$, is eaſily applicable to others.

In general, we conclude that "the greateſt negative coefficient in any equation increaſed by unit, is always a limit that exceeds all the roots of that equation."

But it is to be obſerved at the ſame time, that the greateſt negative coefficient increaſed by unit, is very ſeldom the *neareſt* limit: that is beſt diſcovered by the Rule in the 39th article.

§ 44.

§ 44. Having ſhewn in § 41. how to change any propoſed equation into one that ſhall have all its roots affirmative; we ſhall only treat of ſuch as have all their roots poſitive, in what remains relating to the limits of equations.

Any ſuch equation may be repreſented by $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d}$, &c. $=0$, whoſe roots are a, b, c, d, &c.

And of all ſuch equations two limits are eaſily diſcovered from what precedes, *viz.* 0, which is leſs than the leaſt, and e, found according to § 39. which ſurpaſſes the greateſt root of the equation.

But beſides theſe, we ſhall now ſhew how "*to find other limits betwixt the roots themſelves.*" And, for this purpoſe, will ſuppoſe a to be the leaſt root, b the ſecond root, c the third, and ſo on; it being arbitrary.

§ 45. If you ſubſtitute 0 in place of the unknown quantity, putting $x=0$, the quantity that will ariſe from that ſuppoſition is the laſt term of the equation, all the others, that involve x, vaniſhing.

If you ſubſtitute for x a quantity leſs than the leaſt root a, the quantity reſulting will have the ſame ſign as the laſt term; that is, will be poſitive or negative according as the equation is of an even or odd number of dimenſions. For all the factors $x-a$, $x-b$, $x-c$, &c. will be negative, and their product will be *poſitive* or *negative* according as their number is *even* or *odd.*

If you ſubſtitute for x a quantity greater than the leaſt root a, but leſs than all the other roots, then the ſign of the quantity reſulting will be contrary to what it was before; becauſe one factor $(x - a)$ becomes now poſitive, all the others remaining negative as before.

If you ſubſtitute for x a quantity greater than the two leaſt roots, but leſs than all the reſt; both the factors $x - a$, $x - b$, become poſitive, and the reſt remain as they were. So that the whole product will have the ſame ſign as the laſt term of the equation. Thus ſucceſſively placing inſtead of x quantities that are limits betwixt the roots of the equation, the quantities that reſult will have alternately the ſigns $+$ and $-$. And, *converſely*, "if you find quantities which ſubſtituted in place of x in the propoſed equation, to give alternately poſitive and negative reſults, thoſe quantities are the limits of that equation."

It is uſeful to obſerve, that, in general, "when, by ſubſtituting any two numbers for x in any equation, the reſults have contrary ſigns, one or more of the roots of the equation muſt be betwixt thoſe numbers." Thus, in the equation $x^3 - 2x^2 - 5 = 0$, if you ſubſtitute 2 and 3 for x, the reſults are -5, $+4$; whence it follows that the roots are betwixt 2 and 3: for when theſe reſults have different ſigns, one or other of the factors which produce the equations muſt have changed its ſign; ſuppoſe it is $x=e$, then it is plain that e muſt be betwixt the numbers ſuppoſed equal to x.

§ 46. Let the cubic equation $x^3 - px^2 + qx - r = 0$ be propoſed, and let it be transformed, by aſſuming $y = x - e$, into the equation

$$\left.\begin{array}{r} y^3 + 3ey^2 + 3e^2y + e^3 \\ - py^2 - 2pey - pe^2 \\ + qy + qe \\ - r \end{array}\right\} = 0.$$

Let us ſuppoſe e equal ſucceſſively to the three values of x, beginning with the leaſt value; and becauſe the laſt term $e^3 - pe^2 + qe - r$ will vaniſh in all theſe ſuppoſitions, the equation will have this form,

$$\left.\begin{array}{r} y^2 + 3ey + 3e^2 \\ - py - 2pe \\ + q \end{array}\right\} = 0;$$

where the laſt term $3e^2 - 2pe + q$ is, from the nature of equations, produced of the remaining values of y, or of the exceſſes of two other values of x above what is ſuppoſed equal to e; ſince always $y = x - e$. Now,

1°. If e be equal to the leaſt value of x, then thoſe two exceſſes being both poſitive, they will give a poſitive product, and conſequently $3e^2 - 2pe + q$ will be, in this caſe, poſitive.

2°.

2°. If e be equal to the ſecond value of x, then, of thoſe two exceſſes one being negative and one poſitive, their product $3e^2 - 2pe + q$, will be negative.

3°. If e be equal to the third and greateſt value of x, then the two exceſſes being both negative, their product $3e^2 - 2pe + q$ is poſitive. Whence,

If in the equation $3e^2 - 2pe + q = 0$, you ſubſtitute ſucceſſively in the place of e, the three roots of the equation $e^3 - pe^2 + qe - r = 0$, the quantities reſulting will ſucceſſively have the ſigns $+$, $-$, $+$; and conſequently the three roots of the cubic equation are the limits of the roots of the equation $3e^2 - 2pe + q = 0$ (by § 45.) That is, the leaſt of the roots of the cubic is leſs than the leaſt of the roots of the other; the ſecond root of the cubic is a limit between the two roots of the other; and the greateſt root of the cubic is the limit that exceeds both the roots of the other.

§ 47. We have demonſtrated that the roots of the cubic equation $e^3 - pe^2 + qe - r = 0$ are limits of the quadratic $3e^2 - 2pe + q$; whence it follows (*converſely*) that the roots of the quadratic $3e^2 - 2pe + q = 0$ are the limits between the firſt and ſecond, and between the ſecond and third roots of the cubic $e^3 - pe^2 + qe - r = 0$. So that if you find the limit that exceeds the greateſt root of the cubic, by § 39. you will have (with 0, which is the limit leſs than any of the roots) four limits for the three roots of the propoſed cubic.

It was demonſtrated in § 35. how the quadratic $3e^2 - 2pe + q$ is deduced from the propoſed cubic $e^3 - pe^2 + qe - r = 0$, *viz.* by multiplying each term by the index of e in it, and then dividing the whole by e; and what we have demonſtrated of cubic equations is eaſily extended to all others; ſo that we conclude, "that the laſt term but one of the transformed equation is the equation for determining the limits of the propoſed equation." Or, that the equation ariſing by multiplying each term by the index of the unknown quantity in it, is the equation whoſe roots give the limits of the propoſed equation; if you add to them the two mentioned in § 44.

§ 48. For the ſame reaſon, it is plain that the root of the ſimple equation $3e - p = 0$; (*i. e.* $\frac{1}{3}p$) is the limit between the two roots of the quadratic $3e^2 - 2pe + q = 0$. And, as $4e^3 - 3pe^2 + 2qe - r = 0$ gives three limits of the equation $e^4 - pe^3 + qe^2 - re + s = 0$, ſo the quadratic $6e^2 - 3pe + q = 0$ gives two limits that are betwixt the roots of the cubic $4e^3 - 3pe^2 + 2qe - r = 0$; and $4e - p = 0$ gives one limit that is betwixt the two roots of the quadratic $6e^2 - 3pe + q = 0$. So that we have a compleat ſeries of theſe equations ariſing from a ſimple equation to the propoſed, each of which determines the limits of the following equation.

§ 49. If two roots in the propoſed equation are equal, then "the limit that ought to be betwixt them muſt, in this caſe, become equal to one of the equal roots themſelves." Which perfectly agrees with what was demonſtrated in the laſt chapter, concerning the Rule for finding the equal roots of equations.

And, the ſame equation that gives the limits, giving alſo one of the equal roots, when two or more are equal, it appears, that "if you ſubſtitute a limit in place of the unknown quantity in an equation, and, inſtead of a poſitive or negative reſult, it be found $= 0$, then you may conclude, that not only the limit itſelf is a root of the equation, but that there are two roots in that equation equal to it and to one another."

§ 50. It having been demonſtrated that the roots of the equation $x^3 - px^2 + qx - r = 0$ are the limits of the roots of the equation $3x^2 - 2px + q = 0$, the three roots of the cubic equation, which ſuppoſe to be a, b, c, ſubſtituted for x in the quadratic $3x^2 - 2px + q$, muſt give the reſults poſitive and negative alternately.

nately. Suppoſe theſe three reſults to be $+N$, $-M$, $+L$; that is, $3a^2-2pa+q=N$, $3b^2-2pb+q=-M$, $3c^2-2pc+q=L$; and ſince $a^3-pa^2+qa-r=0$, and $3a^3-2pa^2+qa=N\times a$, ſubtracting the former multiplied into 3 from the latter, the remainder is $pa^2-2qa+3r=N\times a$. In the ſame manner $pb^2-2qb+3r=-M\times b$, and $pc^2-2qc+3r=+L\times c$. Therefore $px^2-2qx+3r$ is ſuch a quantity that if, for x, you ſubſtitute in it ſucceſſively a, b, c, the reſults will be $+N\times a$, $-M\times b$, $+L\times c$. Whence a, b, c, are the limits of the equation $px^2-2qx+3r=0$ (by § 45.) and, *converſely*, the roots of the equation $px^2-2qx+3r=0$ are limits between the firſt and ſecond, and between the ſecond and third roots of the cubic $x^3-px^2+qx-r=0$. Now the equation $px^2-2qx+3r=0$ ariſes from the propoſed cubic by multiplying the terms of this latter by the arithmetical progreſſion 0, −1, −2, −3. And in the ſame manner it may be ſhewn that the roots of the equation $px^3-2qx^2+3rx-4s=0$ are limits of the equation $x^4-px^3+qx^2-rx+s=0$.

Or, multiply the terms of the equation

$$x^3-px^2+qx-r=0$$

by $\underline{a+3b,\ a+2b,\ a+b,\ a,}$

$$\begin{array}{l} ax^3-apx^2+aqx-ar\ (=0) \\ +3bx^3-2bpx^2+bqx\left(=\overline{3x^2-2px+q}\times bx\right). \end{array}$$

Any arithmetical ſeries where a is the leaſt term and b the common difference, and the products (if you ſubſtitute for x, ſucceſſively, a, b, c, the three roots of the propoſed cubic) ſhall be $+N\times bx$, $-M\times bx$, $+L\times bx$. For the firſt part of the product $a\times\overline{x^3-px^2+qx-r}=0$; and a, b, c, being limits in the equation $3x^2-2px+q=0$, their ſubſtitution muſt give reſults N, M, L, alternately poſitive and negative.

In general, the roots of the equation $x^n-px^{n-1}+qx^{n-2}-rx^{n-3}+$, *&c.* $=0$ are limits of the roots of the equation $nx^{n-1}-\overline{n-1}\times px^{n-2}+\overline{n-2}\times qx^{n-3}-\overline{n-3}\times rx^{n-4}+$, *&c.* $=0$; or of any equation that is deduced from it by multiplying its terms by any arithmetical progreſſion $a\mp b$, $a\mp 2b$, $a\mp 3b$, $a\mp 4b$, &c. *And converſely*, the roots of this new equation will be limits of the propoſed equation.

$$x^n-px^{n-1}+qx^{n-2}-, \text{\&c.}=0.$$

"*If any roots of the equation of the limits are impoſſible, then muſt there be ſome roots of the propoſed equation impoſſible.*" For as (in § 46.) the quantity $3e^2-2pe+q$ was demonſtrated to be equal to the product of the exceſſes of two values of x above the third ſuppoſed equal to e; if any impoſſible expreſſion be found in thoſe exceſſes, then there will of conſequence be found impoſſible expreſſions in theſe two values of x.

And “from this obſervation rules may be deduced for diſcovering when there are impoſſible roots in equations.” Of which we ſhall treat afterwards.

§ 51. Beſides the method already explained, there are others by which limits may be determined, which the root of an equation cannot exceed.

Since the ſquares of all real quantities are affirmative, it follows, that “*the ſum of the ſquares of the roots of any equation muſt be greater than the ſquare of the greateſt root.*” And the ſquare root of that ſum will therefore be a limit that muſt exceed the greateſt root of the equation.

If the equation propoſed is $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} +$, &c. $= 0$, then the ſum of the ſquares of the roots (by § 15.) will be $p^2 - 2q$. So that $\sqrt[2]{p^2 - 2q}$ will exceed the greateſt root of that equation.

Or if you find, by § 16. the ſum of the 4th powers of the roots of the equation, and extract the biquadratic root of that ſum, it will alſo exceed the greateſt root of the equation.

§ 52. If you find a mean proportional between the ſum of the ſquares of any two roots, a, b, and the ſum of their biquadrates $(a^4 + b^4)$, this mean proportional will be $\sqrt[2]{a^6 + a^2b^4 + a^4b^2 + b^6}$. And the ſum of the cubes is a^3+b^3. Now ſince $a^2-2ab+b^2$ is the ſquare of $a-b$, it muſt be always poſitive; and if you multiply it by a^2b^2, the product $a^4b^2-2a^3b^3+a^2b^4$ will alſo be poſitive; and conſequently $a^4b^2+a^2b^4$ will be always greater than $2a^3b^3$. Add a^6+b^6, and we have $a^6+a^4b^2+a^2b^4+b^6$ greater than $a^6+2a^3b^3+b^6$; and extracting the root $\sqrt[2]{a^6+a^4b^2+a^2b^4+b^6}$ greater than a^3+b^3. And the ſame may be demonſtrated of any number of roots whatever.

Now if you add the ſum of all the cubes taken *affirmatively* to their ſum with their *proper* ſigns, they will give double the ſum of the cubes of the affirmative roots. And if you ſubtract the ſecond ſum from the firſt, there will remain double the ſum of the cubes of the negative roots. Whence it follows, that "half the ſum of the mean proportional betwixt the ſum of the ſquares and the ſum of the biquadrates, and of the ſum of the cubes of the roots with their proper ſigns, exceeds the ſum of the cubes of the affirmative roots:" and "half their difference exceeds the ſum of the cubes of the negative roots." And by extracting the cube root of that ſum and difference, you will obtain limits that ſhall exceed the ſums of the affirmative and of the negative roots. And ſince it is eaſy, from what has been already explained, to diminiſh the roots of an equation ſo that they all may become negative but one, it appears how by this means you may approximate very near to that root. But this does not ſerve when there are impoſſible roots.

Several other Rules like theſe might be given for limiting the roots of equations. We ſhall give one not mentioned by other Authors.

In a cubic $x^3-px^2+qx-r=0$ find q^2-2pr, and call it e^4; then ſhall the greateſt root of the equation always be greater than $\frac{e}{\sqrt[4]{3}}$, or $\sqrt[4]{\frac{e^4}{3}}$. And,

In any equation $x^n-px^{n-1}+qx^{n-2}-rx^{n-3}$ $+$, &c. $=0$ find $\frac{q^2-2pr+2s}{n}$, and extracting the root of the fourth power out of that quantity, it ſhall always be leſs than the greateſt root of the equation.

CHAP. VI.

Of the Reſolution of Equations, all whoſe Roots are commenſurate.

§ 53. IT was demonſtrated, in *Chap.* 2. that the laſt term of any equation is the product of its roots: from which it follows, that the roots of an equation, when commenſurable quantities, will be found among the diviſors of the laſt term. And hence we have for the reſolution of equations this

RULE.

Bring all the terms to one ſide of the equation, find all the diviſors of the laſt term, and ſubſtitute them ſucceſſively for the unknown quantity in the equation. So ſhall that diviſor which, ſubſtituted in this manner, gives the reſult $=0$, *be the root of the propoſed equation.*

For example, ſuppoſe this equation is to be reſolved,

$$\left.\begin{array}{l} x^3 - 3ax^2 + 2a^2x - 2a^2b \\ \quad\; - bx^2 + 3abx \end{array}\right\} = 0,$$

where the laſt term is $2a^2b$, whoſe ſimple literal diviſors are a, b, $2a$, $2b$, each of which may be taken either poſitively or negatively: but as here we find there are variations of ſigns in the equation, we need only take them poſitively. Suppoſe $x=a$ the firſt of the diviſors, and ſubſtituting a for x, the equation becomes

$$\left.\begin{array}{l} a^3-3a^3+2a^3-2a^2b \\ -a^2b+3a^2b \end{array}\right\} \text{or, } 3a^3-3a^3+3a^2b-3a^2b=0.$$

So that, the whole vaniſhing, it follows that a is one of the roots of the equation.

After the ſame manner, if you ſubſtitute b in place of x, the equation is

$$\left.\begin{array}{l} b^3-3ab^2+2a^2b-2a^2b \\ -b^3+3ab^2 \end{array}\right\}=0,$$

which vaniſhing ſhews b to be another root of the equation.

Again, if you ſubſtitute $2a$ for x, you will find all the terms deſtroy one another ſo as to make the ſum $=0$. For it will then be

$$\left.\begin{array}{l} 8a^3-12a^3+4a^3-2a^2b \\ -4a^2b+6a^2b \end{array}\right\}=0.$$

Whence we find that $2a$ is the third root of the equation. Which, after the firſt two ($+a$, $+b$) had been found, might have been collected from this, that the laſt term being the product of the three roots, $+a$, $+b$ being known, the third muſt neceſſarily be equal to the laſt term divided by the product ab, that is, $= \frac{2a^2b}{ab} = 2a$.

Let the roots of the cubic equation

$x^3 - 2x^2 - 33x + 90 = 0$ be required.

And firſt the diviſors of 90 are found to be 1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90. If you ſubſtitute 1 for x, you will find $x^3 - 2x^2 - 33x + 90 = 56$; ſo that 1 is not a root of the equation. If you ſubſtitute 2 for x, the reſult will be 24: but putting $x = 3$, you have

$x^3 - 2x^2 - 33x + 90 = 27 - 18 - 99 + 90 = 117 - 117 = 0$.

So that 3 is one of the roots of the propoſed equation. The other affirmative root is $+5$; and after you find it, as it is manifeſt from the equation, that the other root is negative, you are not to try any more diviſors taken poſitively, but to ſubſtitute them, negatively taken, for x: and thus you find that -6 is the third root. For putting $x = -6$, you have

$x^3 - 2x^2 - 33x + 90 = -216 - 72 + 198 + 90 = 0$.

This laſt root might have been found by dividing the laſt term 90, having its ſign changed, by 15, the product of the two roots already found.

§ 55. When one of the roots of an equation is found, in order to find the reſt with leſs trouble, divide the propoſed equation by the ſimple equation which you are to deduce from the root already found, and the quotient ſhall give an equation of a degree lower than the propoſed; whoſe roots will give the remaining roots required.

As for example, the root $+3$, firſt found, gave $x=3$ or $x-3=0$, whence dividing thus,

$$\begin{array}{l}
x-3)\ x^3-2x^2-33x+90\ (x^2+x-30 \\
\qquad\ \ x^3-3x^2 \\
\hline
\qquad\qquad\quad x^2-33x+90 \\
\qquad\qquad\quad x^2-\ \ 3x \\
\hline
\qquad\qquad\qquad\ -30x+90 \\
\qquad\qquad\qquad\ -30x+90 \\
\hline
\qquad\qquad\qquad\quad\ 0\quad\ \ 0
\end{array}$$

The quotient ſhall give a quadratic equation $x^2+x-30=0$, which muſt be the product of the other two ſimple equations from which the cubic is generated, and whoſe roots therefore muſt be two of the roots of that cubic.

Now the roots of that quadratic equation are eaſily found by *Chap.* 13. *Part* I. to be $+5$ and -6. For,

$$x^2+x=30;$$

add $\frac{1}{4}$ . . . $x^2+x+\frac{1}{4}=30+\frac{1}{4}=\frac{121}{4}$,

$\surd$ . . . . . $x+\frac{1}{2}=\pm\sqrt{\frac{121}{4}}=\pm\frac{11}{2}$,

and . . . . $x=\pm\frac{11}{2}-\frac{1}{2}=+5$, or -6.

§ 56.

§ 56. After the ſame manner, if the biquadratic $x^4-2x^3-25x^2+26x+120=0$ is to be reſolved; by ſubſtituting the diviſors of 120 for x, you will find that $+3$, one of thoſe diviſors, is one of the roots; the ſubſtitution of 3 for x giving $81-54-225+78+120=279-279=0$. And therefore dividing the propoſed equation by $x-3$, you muſt enquire for the roots of the cubic $x^3+x^2-22x-40=0$, and finding that $+5$, one of the diviſors of 40, is one of the roots, you divide that cubic by $x-5$, and the quotient gives the quadratic $x^2+6x+8=0$, whoſe two roots are -2, -4. So that the four roots of the biquadratic are $+3$, $+5$, -2, -4.

§ 57. This Rule ſuppoſes that you can find all the diviſors of the laſt term; which you may always do thus.

> " *If it is a ſimple quantity, divide it by its leaſt diviſor that exceeds unit, and the quotient again by its leaſt diviſor, proceeding thus till you have a quotient that is not diviſible by any number greater than unit.* This quotient, with theſe diviſors, are the firſt or *ſimple diviſors* of the quantity. And the products of the multiplication of any 2, 3, 4, *&c.* of them are the *compound diviſors*."

As, to find the diviſors of 60; firſt I divide by 2, and the quotient 30 again by 2, then the next quotient 15 by 3, and the quotient of this diviſion 5 is not farther diviſible by any integer above units; ſo that

The ſimple diviſors are . . . . . . 2, 2, 3, 5.
The products of two, ... 4, 6, 10, 15.
The products of three, ... 12, 20, 30.
The product of all four, ... 60.

The diviſors of 90 are found after the ſame manner.

Simple diviſors, ... 2, 3, 3, 5.
The products of two, ... 6, 9, 10, 15.
The products of three, ... 18, 30, 45.
The product of all four, ... 90.

The diviſors of $21abb$.

The ſimple diviſors, ... $3, 7, a, b, b$.
The products of two, $21, 3a, 3b, 7a, 7b, ab, bb$.
The products of three, $21a, 21b, 3ab, 3bb$.
$7ab, 7bb, abb$.
The products of four, $21ab, 21bb, 3abb, 7abb$.
The product of the five, ... $21ab$.

§ 58. But as the laſt term may have very many diviſors, and the labour may be very great to ſubſtitute them all for the unknown quantity, we ſhall now ſhow how it may be abridged, by limiting to a ſmall number the diviſors you are to try. And firſt it is plain, from § 42. that "any diviſor that exceeds the greateſt negative coefficient by unity is to be neglected." Thus in reſolving the equation $x^4-2x^3-25x^2+26x+120=0$, as 25 is the greateſt negative coefficient, we conclude that the diviſors of 120 that exceed 26 may be neglected.

But the labour may be ſtill abridged, if we make uſe of the Rule in § 39; that is, if we find the number, which ſubſtituted in theſe following expreſſions,

$$\begin{aligned}&x^4-2x^3-25x^2+26x+120\\&2x^3-3x^2-25x+13,\\&6x^2-6x-25,\\&2x-1,\end{aligned}$$

will give in them all a poſitive reſult: for that number will be greater than the greateſt root, and all the diviſors of 120 that exceed it may be neglected.

That this inveſtigation may be eaſier, we ought to begin always with that expreſſion, where the negative roots ſeem to prevail moſt; as here in the quadratic expreſſion $6x^2-6x-25$; where finding that 6 ſubſtituted for x gives that expreſſion poſitive, and gives all the other expreſſions at the ſame time poſitive, I conclude that 6 is greater than any of the roots, and that all the diviſors of 120 that exceed 6 may be neglected.

If the equation $x^3+11x^2+10x-72=0$ is propoſed, the Rule of § 42 does not help to abridge the operation; the laſt term itſelf being the greateſt negative term. But, by § 39. we enquire what number ſubſtituted for x will give all theſe expreſſions poſitive:

$$x^3+11x^2+10x-72,$$
$$3x^2+22x+10,$$
$$3x+11.$$

Where the labour is very ſhort, ſince we need only attend to the firſt expreſſion; and we ſee immediately that 4 ſubſtituted for x gives a poſitive reſult, whence all the diviſors of 72 that exceed 4 are to be rejected; and thus by a few trials we find that $+2$ is the poſitive root of the equation. Then dividing the equation by $x-2$, and reſolving the quadratic equation that is the quotient of the diviſion, you find the other two roots to be -9, and -4.

§ 59. But there is another method that reduces the diviſors of the laſt term, that can be uſeful, ſtill to more narrow limits.

Suppoſe the cubic equation $x^3-px^2+qx-r=0$ is propoſed to be reſolved. Transform it to an equation whoſe roots ſhall be leſs than the values of x by unity, aſſuming $y=x-1$. And the laſt term of the transformed equation will be $1-p+q-r$; which is found by ſubſtituting unit, the difference of x and y, for x, in the propoſed equation; as will eaſily appear from § 24. where, when $y=x-e$, the laſt term of the transformed equation was e^3-pe^2+qe-r.

Transform again the equation $x^3-px^2+qx-r=0$, by aſſuming $y=x+1$, into an equation whoſe roots ſhall exceed the values of x by unit, and the laſt term of the transformed equation will be $-1-p-q-r$, the ſame that ariſes by ſubſtituting -1, the difference betwixt x and y, for x, in the propoſed equation.

Now the values of x are ſome of the diviſors of r, which is the term left when you ſuppoſe $x=0$; and the values of the y's are ſome of the diviſors of $+1-p+q-r$, and of $-1-p-q-r$, reſpectively. And theſe values are in arithmetical progreſſion increaſing by the common difference unit; becauſe $x-1$, x, $x+1$, are in that progreſſion. And it is obvious the ſame reaſoning may be extended to any equation of whatever degree. So that this gives a general method for the reſolution of equations whoſe roots are commenſurable.

RULE.

" *Subſtitute in place of the unknown quantity ſucceſſively the terms of the progreſſion* 1, 0, −1, &c. *and find all the diviſors of the ſums that reſult; then take out all the arithmetical progreſſions you can find among theſe diviſors, whoſe*

whoſe common difference is unit; and the values of x will be among the diviſors ariſing from the ſubſtitutions of $x=0$ that belong to theſe progreſſions." The values of x will be affirmative when the arithmetical progreſſion increaſes, but negative when it decreaſes.

EXAMPLE.

§ 60. Let it be required to find one of the roots of the equation $x^3-x^2-10x+6=0$. The operation is thus:

Suppoſit.	*Reſult.*		*Diviſors.*	*Arith. prog. decr.*
$x=1$	. . . .	-4	1, 2, 4	4
$x=0$	$x^3-x^2-10x+6=$	$+6$	1, 2, 3, 6	3 gives $x=-3$
$x=-1$	. . . .	$+14$	1, 2, 7, 14	2

Where the ſuppoſitions of $x=1$, $x=0$, $x=-1$ give the quantity $x^3-x^2-10x+6$ equal to -4, 6, 14; among whoſe diviſors we find only one arithmetical progreſſion, 4, 3, 2; the term of which oppoſite to the ſuppoſition of $x=0$, being 3, and the ſeries decreaſing, we try if -3 ſubſtituted for x makes the equation vaniſh; which ſucceeding one of its roots muſt be -3. Then dividing the equation by $x+3$, we find the roots of the (quadratic) quotient

$$x^2-4x+2=0 \text{ are } 2\pm\sqrt{2}.$$

§ 61. If it is required to find the roots of the equation $x^3-3x^2-46x-72=0$, the operation will be thus:

Suppoſ.	*Reſults*	*Diviſors.*	*Progreſſions.*			
$x = 1$	— 120	1,2,3,4,5,6,8,10,12,15,20,24,30,40,60,120.	8	3	4	5
$x = 0$	— 72	1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72.	9	2	3	4
$x = -1$	— 30	1, 2, 3, 5, 6, 10, 15, 30.	10	1	2	3

Of theſe four arithmetical progreſſions having their common difference equal to unit, the
firſt

firſt gives $x=9$, the others give $x=-2$, $x=-3$, $x=-4$; all which ſucceed except $x=-3$: ſo that the three values of x are $+9$, -2, -4.

CHAP. VII.

Of the Reſolution of Equations by finding the equations of a lower degree that are their diviſors.

§ 62. TO find the roots of an equation is the ſame thing as to find the *ſimple* equations, by the multiplication of which into one another it is produced, or to find the ſimple equations that divide it without a remainder.

If ſuch ſimple equations cannot be found, yet if we can find the *quadratic* equations from which the propoſed equation is produced, we may diſcover its roots afterwards by the reſolution of theſe quadratic equations. Or, if neither theſe ſimple equations nor theſe quadratic equations can be found, yet, by finding a *cubic* or *biquadratic* that is a diviſor of the propoſed equation, we may depreſs it lower, and make the ſolution more eaſy.

Now

Now, in order to find the Rules by which theſe diviſors may be diſcovered, we ſhall ſuppoſe that

$$\left.\begin{array}{l} mx-n \\ mx^2-nx+r \\ mx^3-nx^2+rx-s \end{array}\right\}\text{are the}\left\{\begin{array}{l}\text{ſimple}\\ \text{quadratic}\\ \text{cubic}\end{array}\right.$$

diviſors of the propoſed equation; and if E repreſent the quotient ariſing by dividing the propoſed equation by that diviſor, then

$E\times\overline{mx-n}$,

$E\times\overline{mx^2-nx+r}$,

or, $E\times\overline{mx^3-nx^2+rx-s}$, will repreſent the propoſed equation itſelf. Where it is plain, that "ſince m is the coefficient of the higheſt term of the diviſors, it muſt be a diviſor of the coefficient of the higheſt term of the propoſed equation."

§ 63. Next we are to obſerve, that, ſuppoſing the equation has a ſimple diviſors $mx-n$, if we ſubſtitute in the equation $E\times\overline{mx-n}$, in place of x, any quantity, as a, then the quantity that will reſult from this ſubſtitution will neceſſarily have $ma-n$ for one of its diviſors; ſince, in this ſubſtitution, $mx-n$ becomes $ma-n$.

If we ſubſtitute ſucceſſively for x any arithmetical progreſſion, a, $a-e$, $a-2e$, &c. the quantities that will reſult from theſe ſubſtitutions, will have among their diviſors

$ma - n$,

$ma - me - n$,

$ma - 2me - n$, which are alſo in arithmetical progreſſion, having their common difference equal to me.

If, for example, we ſubſtitute for x the terms of this progreſſion, $1, 0, -1$, the quantities that reſult have among their diviſors the arithmetical progreſſion $m - n, -n, -m - n$; or, changing the ſigns, $n - m, n, n + m$. Where the difference of the terms is m, and the term belonging to the ſuppoſition of $x = 0$ is n.

§ 64. It is manifeſt therefore, that when an equation has any ſimple diviſor, if you ſubſtitute for x the progreſſion $1, 0, -1$, there will be found amongſt the diviſors of the ſums that reſult from theſe ſubſtitutions, one arithmetical progreſſion at leaſt, whoſe common difference will be unit or a diviſor m of the coefficient of the higheſt term, and which will be the coefficient of x in the ſimple diviſor required: and whoſe term, ariſing from the ſuppoſition of $x = 0$, will be n the other member of the ſimple diviſor $mx - n$.

From which this Rule is deduced for diſcovering ſuch a ſimple diviſor, when there is any.

RULE.

"*Subſtitute for* x *in the propoſed equation ſucceſſively the numbers* 1, 0, -1. *Find all the diviſors of the ſums that reſult from this ſubſtitution, and take out all the arithmetical progreſſions you can find amongſt them, whoſe difference is unit, or ſome diviſor of the coefficient of the higheſt term of the equation. Then ſuppoſe* n *equal to that term of any one progreſſion that ariſes from the ſuppoſition of* $x = 0$, *and* $m =$ *the foreſaid diviſor of the coefficient of the higheſt term of the equation, which* m *is alſo the difference of the terms of this progreſſion; ſo ſhall you have* $mx - n$ *for the diviſor required.*"

You may find arithmetical progreſſions giving diviſors that will not ſucceed; but if there is any diviſor, it will be found thus by means of theſe arithmetical progreſſions.

§ 65. If the equation propoſed has the coefficient of its higheſt term 1, then it will be $m = 1$, and the diviſor will be $x - n$, and the rule will coincide with that given in the end of the laſt chapter, which we demonſtrated after a different manner; for the diviſor being $x - n$, the value of x will be $+n$, the term of the progreſſion that is a diviſor of the ſum that ariſes from ſuppoſing $x = 0$. Of this caſe we gave examples in the laſt chapter; and though it it eaſy to reduce an equation whoſe higheſt term has a coefficient different from unit, to one where that coefficient ſhall be unit, by § 30; yet, without that reduction, the equation may be reſolved by this rule, as in the following

EXAMPLE.

§ 66. Suppoſe $8x^3-26x^2+11x+10=0$, and that it is required to find the values of x; the operation is thus:

Suppoſ.	*Reſults.*		*Diviſors.*	*Progr.*	
$x=1$	$8x^3-26x^2+11x+10=$	$+3$	1, 3.	3	3
$x=0$		$+10$	1, 2, 5, 10.	2	5
$x=-1$		-35	1, 5, 7, 35.	1	7

The difference of the terms of the laſt arithmetical progreſſion is 2, a diviſor of 8, the coefficient of the higheſt term x^3 of the equation, therefore ſuppoſing $m=2$, $n=5$, we try the diviſor $2x-5$; which ſucceeding, it follows that $1x-5=0$, or $x=2\frac{1}{2}$.

The quotient is the quadratic $4x^2-3x-2=0$, whoſe roots are $\frac{3+\sqrt{41}}{8}$, and $\frac{3-\sqrt{41}}{8}$, ſo that the three roots of the propoſed equation are $2\frac{1}{2}$, $\frac{3+\sqrt{41}}{8}$, $\frac{3-\sqrt{41}}{8}$. The other arithmetical progreſſion gives $x+2$, for a diviſor; but it does not ſucceed.

§ 67. If the propoſed equation has no ſimple diviſor, then we are to enquire if it has not ſome quadratic diviſor (if itſelf is an equation of more than three dimenſions).

An equation having the diviſor mx^2-nx+r may be expreſſed as in the firſt article of this chapter, by $E\times\overline{mx^2-nx+r}$; and if we ſubſtitute for x any known quantity a, the ſum that will reſult will have ma^2-na+r for one of its diviſors; and, if we ſubſtitute ſucceſſively for x the progreſſion a, $a-e$, $a-2e$, $a-3e$, &c. the ſums that ariſe from this ſubſtitution will have

$$\begin{array}{l} ma^2-na+r, \\ m\times\overline{a-e}|^2-n\times\overline{a-e}+r, \\ m\times\overline{a-2e}|^2-n\times\overline{a-2e}+r, \\ m\times\overline{a-3e}|^2-n\times\overline{a-3e}+r, \\ \text{\&c.} \end{array}$$

amongſt their diviſors, reſpectively.

Theſe terms are not now, as in the laſt caſe, in arithmetical progreſſion; but if you ſubtract them from the ſquares of the terms a, $a-e$, $a-2e$, $a-3e$, &c. multiplied by m a diviſor of the higheſt term of the propoſed equation, that is from

ma^2,

$m \times \overline{a-e}|^2$,

$m \times \overline{a-2e}|^2$,

$m \times \overline{a-3a}|^2$, &c. the remainders,

$na-r$,

$n \times \overline{a-e}-r$,

$n \times \overline{a-2e}-r$,

$n \times \overline{a-3e}-r$, &c. ſhall be in arithmetical progreſſion, having their common difference equal to $n \times e$.

If, for example, we ſuppoſe the aſſumed progreſſion a, $a-e$, $a-2e$, $a-3e$, &c. to be 2, 1, 0, −1, the diviſors will be

$$\left.\begin{array}{r} 4m-2n+r, \\ m-\ n+r, \\ +r, \\ m+n+r, \end{array}\right\} \begin{array}{c}\text{which} \\ \text{ſubtracted} \\ \text{from}\end{array} \left\{\begin{array}{c} 4m \\ m \\ 0 \\ m \end{array}\right\} \text{reſpectively,}$$

leave $2n-r$,

$n-r$,

$-r$,

$-n-r$, an arithmetical progreſſion, whoſe difference is $+n$; and whoſe term ariſing from the ſubſtitution of 0 for x is $-r$.

From which it follows, that by this operation, if the propoſed equation has a quadratic diviſor, you will find an arithmetical progreſſion that will determine to you n and r, the coefficient m being ſuppoſed known; ſince it is unit, or a diviſor of the coefficient of the higheſt term of the equation. Only you are to obſerve, that if the firſt term mx^2 of the quadratic diviſor is negative, then in order to obtain an arithmetical progreſſion, you are not to ſubtract, but add the diviſors $-4m-2n+r$, $-m-n+r$, $+r$, $-m+n+r$, to the terms $4m$, m, 0, m.

§ 68. The general Rule therefore, deduced from what we have ſaid, is,

“ *Subſtitute in the propoſed equation for* x *the terms* 2, 1, 0, −1, &c. *ſucceſſively. Find all the diviſors of the ſums that reſult, adding and ſubtracting them from the ſquares of theſe numbers,* 2, 1, 0, −1, &c. *multiplied by a numerical diviſor of the higheſt term of the propoſed equation, and take out all the arithmetical progreſſions that can be found amongſt theſe ſums and differences. Let* r *be that term in any progreſſion that ariſes from the ſubſtitution of* $x=0$, *and let* $\mp n$ *be the difference ariſing from ſubtracting that term from the preceding term in the progreſſion; laſtly, let* m *be the foreſaid diviſor of the higheſt term; then ſhall* $\overline{mx^2 \pm nx - r}$ *be the diviſor that ought to be tried.*” And one or other of the diviſors found in this manner will ſucceed, if the propoſed equation has a quadratic diviſor.

§ 69. Suppoſe, for example, the biquadratic $x^4 - 5x^3 + 7x^2 - 5x - 6 = 0$ is propoſed, which has no ſimple diviſor; then to diſcover if it has any quadratic diviſor, the operation is thus;

Suppoſ.	*Reſults.*	*Diviſors.*	*Squares.*	*Sums and differences of the diviſors and ſquares.*	*Arithmetical Progreſſions.*			
$x = 2$	-12	1, 2, 3, 4, 6, 12.	4	−8, −2, 0, 1, 2, 3, 5, 6, 7, 8, 10, 16.	8	1	−8	3
$x = 1$	-8	1, 2, 4, 8.	1	−7, −3, −1, 0, 2, 3, 5, 9.	5	−1	−7	2
$x = 0$	-6	1, 2, 3, 6,	0	−6, −3, −2, −1, 1, 2, 3, 6.	2	−3	−6	1
$x = -1$	$+12$	1, 2, 3, 4, 6, 12.	1	−11, −5, −3, −2, −1, 0, 2, 3, 4, 5, 7, 13.	−1	−5	−5	0

The firſt arithmetical progreſſion gives the diviſor x^2-3x-2; the ſecond gives x^2-2x+3: both which ſucceed, ſo that the roots of the two equations $x^2-3x-2=0$, and $x^2-2x+3=0$, *viz.* $\frac{3\pm\sqrt{17}}{2}$ and $1\pm\sqrt{-2}$, are the four roots of the propoſed equation, the two laſt of which are impoſſible. The diviſors which the other arithmetical progreſſions give, do not ſucceed.

§ 70. After the ſame manner a Rule may be diſcovered for finding the cubic diviſors, or thoſe of higher dimenſions, of any propoſed equation.

Suppoſe the cubic diviſor to be mx^3-nx^2+rx-s, and by ſuppoſing x equal to the terms of the arithmetical progreſſion, it will be as follows:

Suppoſ.	*Reſults.*	*Cubes of terms of progr. xm.*	*1ſt Differ.*	*2d Diff.*	*3d Diff.*
$x=3$	$27m-9n+3r-s$	$27m$	$9n-3r+s$	$5n-r$	$2n$
$x=2$	$8m-4n+2r-s$	$8m$	$4n-2r+s$	$3n-r$	$2n$
$x=1$	$m-n+r-s$	m	$n-r+s$	$n-r$	$2n$
$x=0$	$-s$	0	$+s$	$-n-r$	
$x=-1$	$-m-n-r-s$	$-m$	$n+r+s$		

Where the firſt differences are not themſelves in arithmetical progreſſion, as in the laſt caſe, but the differences of its terms, or the ſecond differences, are in arithmetical progreſſion, the common difference being $2n$, whence n is known. The quantity r is found in the column of the ſecond difference, and s is always to be aſſumed ſome diviſor of the laſt term of the propoſed equation, as m is of the coefficient of the firſt term. Whence all the coefficients of a diviſor $mx^3 - nx^2 + rx - s$, with which trial is to be made, may be determined.

If it is a diviſor of four dimenſions that is required, by proceeding in like manner, you will obtain a ſeries of differences whoſe ſecond differences are in arithmetical progreſſion. If it is a diviſor of five dimenſions that is required, you will obtain, in the ſame manner, a progreſſion whoſe third differences will be in arithmetical progreſſion; and by obſerving theſe progreſſions, you may diſcover rules for determining the coefficients of the diviſor required.

The foundation of theſe Rules being, that, if an arithmetical progreſſion a, $a+e$, $a+2e$, $a+3e$, &c. is aſſumed, the firſt differences of their ſquares will be in arithmetical progreſſion; thoſe differences being $2ae+e^2$, $2ae+3e^2$, $2ae+5e^2$, &c. whoſe common difference is $2e^2$ And the ſecond differences of their cubes, and the third differences of their fourth powers are likewiſe in arithmetical progreſſion, as is eaſily demonſtrated.

§ 71. Hitherto we have only ſhewn how to find the diviſors of equations that involve but *one letter*. But the ſame rules ſerve for diſcovering the diviſors when there are two letters, if all the terms have the ſame dimenſions; for, "*by ſuppoſing one of the letters equal to unit, find the diviſor by the preceding Rules, and then by compleating the dimenſions of the diviſor, ſubſtituting the letter again for unit, you will have the diviſor required.*"

Suppoſe, for example, you are to find the diviſor of $8x^3-26ax^2+11a^2x+10a^3=0$, by putting $a=1$, that quantity becomes $8x^3-26x^2+11x+10=0$; whoſe diviſor was found, § 66. to be $2x-5$; now multiply the term -5 by $+a$, to bring it to the ſame dimenſions as the other, and the diviſor required is $2x-5a$.

§ 72. Beſides the method hitherto explained for finding the diviſors of lower dimenſions that may divide the propoſed equation, there are others that deſerve to be conſidered. The following is applicable to equations of all ſorts, though we give it only for thoſe of four dimenſions.

Let the biquadratic $x^4-px^3+qx^2-rx+s=0$ be the equation propoſed; and let us ſuppoſe it is the product of theſe two quadratic equations,

$$\begin{array}{l} x^2 - mx + n = 0 \\ \times\, x^2 - kx + l = 0 \\ \hline \end{array}$$

$$x^4 \left.\begin{array}{l} -k \\ -m \end{array}\right\} x^3 \left.\begin{array}{l} +l \\ +n \\ +mk \end{array}\right\} x^2 \left.\begin{array}{l} -ml \\ -nk \end{array}\right\} x + ln = 0,$$

the terms of which will be equal, reſpectively, to the terms of the propoſed equation.

In this equation, l and n being diviſors of the laſt term s, we may conſider one of them (*viz.* l) as known; and in order to find m or k, we need only compare the terms of this equation with the terms of the propoſed equation reſpectively, which gives,

$$\begin{array}{ll} 1^\circ. & k + m = p. \\ 2^\circ. & mk + l + n = q. \\ 3^\circ. & ml + nk = r. \\ 4^\circ. & nl = s. \end{array}$$

Now in order to find an equation that ſhall involve only k, and known terms, take the two values of m that ariſe from the firſt and third equations, and you will find,

$m = p - k = \frac{r - nk}{l}$ (becauſe $n = \frac{s}{l}$, by equation the fourth) $= \frac{r - \frac{ks}{l}}{l} = \frac{rl - ks}{l^2}$; whence $pl^2 - kl^2 = rl - ks$, and $k = \frac{pl^2 - rl}{l^2 - s}$; and the quadratic $x^2 - kx + l = 0$ becomes $x^2 - \frac{pl^2 - rl}{l^2 - s} \times x + l = 0$.

To apply this to practice, you muſt ſubſtitute ſucceſſively for l all the diviſors of s, the laſt term of the propoſed equation, till you find one of them ſuch, that $x^2 - \frac{pl^2 - rl}{l^2 - s} \times x + l$ can divide the propoſed equation without a remainder.

EXAMPLE

§ 73. If the equation $x^4 - 6x^3 + 20x^2 - 34x + 35 = 0$ is propoſed. The diviſors of 35 are 1, 5, 7, 35; if you put $l = 1$, the quadratic that ariſes will not ſucceed. But if you ſuppoſe $l = 5$, then the equation $x^2 - kx + l$, that is $x^2 - \frac{pl^2 - rl}{l^2 - s} \times x + l = 0$ becomes

$$x^2 - \frac{6 \times 25 - 34 \times 5}{25 - 35} x + 5 = 0 = x^2 - 2x + 5,$$

which divides the propoſed equation without a remainder, and gives the quotient $x^2 - 4x + 7 = 0$.

"In this operation it is unneceſſary to try any diviſor l, that exceeds the ſquare root of s, the laſt term of the propoſed equation." And, if the propoſed equation is literal, "you need only try thoſe diviſors of the laſt term that are of two dimenſions."

If, in any ſuppoſition of l, the value of k, viz. $\frac{pl^2 - rl}{l^2 - s}$, becomes a fraction, then that ſuppoſition is to be rejected, and another value of l to be tried.

§ 74. By comparing the ſecond and fourth equations of the laſt article, you may obtain another value of k. For $n = q - l - mk = \frac{s}{l}$; ſo that ($m$ being equal to $p - k$) $\frac{s}{l} = q - l - pk + k^2$, and $k^2 - pk + q - l - \frac{s}{l} = 0$. Which gives $k = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - q + l + \frac{s}{l}}$. So that the quadratic diviſor required becomes

$$x^2 - \overline{\tfrac{1}{2}p \mp \sqrt{\tfrac{1}{4}p^2 - q + l + \tfrac{s}{l}}} \times x + l = 0.$$

This diviſor muſt be tried when $l = \frac{s}{l}$, and at the ſame time $l = \frac{r}{p}$, the former expreſſion not ſerving in that caſe.

By this formula, diviſors may be found whoſe ſecond terms may be *irrational.*

How the diviſors of higher equations may be found, when they have any, may be underſtood from what has been ſaid of thoſe of four dimenſions.

SUPPLEMENT to CHAP. VII.

Of the Reduction of Equations by Surd diviſors.

AN equation of four, ſix, or more dimenſions, although it may admit of no rational diviſor, may have one that is irrational. As the biquadratic $x^4 + px^3 + qx^2 + rx + s = 0$, which we ſuppoſe to be irreducible by any rational diviſor, may yet, by adding a ſquare $k^2x^2 + 2klx + l^2$ multiplied into ſome quantity n, be compleated into a ſquare $\overline{x^2 + \frac{1}{2}px + \mathcal{Q}}|^2$. In which caſe we ſhall have $x^2 + \frac{1}{2}p + \mathcal{Q} = \sqrt{n} \times \overline{kx + l}$, and x is found by the reſolution of an affected quadratic equation.

To reduce a biquadratic equation in this manner, we have the following

RULE.

[*] If the biquadratic is $x^4 + px^3 + qx^2 + rx + s = 0$, where p, q, r, s, repreſent the given coefficients under their proper ſigns, *put* $q - \frac{1}{4}p^2 = \alpha$, $r - \frac{1}{2}\alpha p = \beta$, $s - \frac{1}{4}\alpha^2 = \zeta$. And for n *take ſome integer common diviſor of* β *and* 2ζ, *that is not a ſquare number, and which, if either* p *or* r *is an odd number, muſt be odd, and, di-*

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[*] *Arithmet. Univerſ.* pag. 264.

vided by 4, *leave the remainder unity. Write likewiſe for k ſome diviſor of* $\frac{\beta}{n}$ *if p is an even number, or the half of an odd diviſor if p is odd, or* 0 *if* $\beta = 0$. *Subtract* $\frac{\beta}{nk}$ *from* $\frac{1}{2}pk$, *and let the remainder be l. For* $\mathcal{Q}$ *put* $\frac{\alpha + nk^2}{2}$, *and try if, dividing* $\mathcal{Q}^2 - s$ *by n, the root of the quotient is rational and equal to l; if it is, add* $nk^2x^2 + 2nklx + nl^2$ *to both ſides of the equation, and extracting the root you ſhall have* $x^2 + \frac{1}{2}px + \mathcal{Q} = n^{\frac{1}{2}} \times \overline{kx + l}$.

EXAMPLE I.

Let the equation propoſed be $x^4 + 12x - 17 = 0$, and becauſe $p = 0$, $q = 0$, $r = 12$, $s = -17$, we ſhall have $\alpha = 0$, $\beta = 12$, $\zeta = -17$. And β and 2ζ, that is 12 and -34, having only 2 for a common diviſor, it muſt be $n = 2$. Again, $\frac{\beta}{n} = 6$, whoſe diviſors 1, 2, 3, 6, are to be ſucceſſively put for k, and -3, $-\frac{3}{2}$, -1, $-\frac{1}{2}$ for l reſpectively.

But $\frac{\alpha + nk^2}{2}$, that is k^2, is equal to $\mathcal{Q}$, and $\sqrt{\frac{\mathcal{Q}^2 - s}{n}} = l$. And when the even diviſors 2 and 6 are ſubſtituted for k, $\mathcal{Q}$ becomes 4 and 36, and $Q^2 - s$ being an odd number, is not diviſible by $n(=2)$. Wherefore 2 and 6 are to be ſet aſide. But when 1 and 3 are written for k, Q is 1 or 9, and $Q^2 - s$ is 18 or 98 reſpectively; which numbers can be divided by 2, and the roots of the quotients extracted, being ± 3 and ± 7; but only one of them, *viz.* -3, coincides with l. I put therefore $k=1$, $l=-3$, $Q=1$, and adding to both ſides of the equation $nk^2x^2 + 2nklx + nl^2$, that is, $2x^2 - 12x + 18$, there reſults $x^4 + 2x^2 + 1 = 2x^2 - 12x + 18$, and extracting the root of each, $x^2 + 1 = \pm \sqrt{2} \times \overline{x-3}$. And again, extracting the root of this laſt, the four values of x, according to the varieties in the ſigns, are

$$-\tfrac{1}{2}\sqrt{2}+\sqrt{3\sqrt{2}-\tfrac{1}{2}},\quad -\tfrac{1}{2}\sqrt{2}-\sqrt{3\sqrt{2}-\tfrac{1}{2}};$$

$$\tfrac{1}{2}\sqrt{2}+\sqrt{-3\sqrt{2}-\tfrac{1}{2}},\quad \tfrac{1}{2}\sqrt{2}-\sqrt{-3\sqrt{2}-\tfrac{1}{2}},$$

being the roots of $x^4 + 12x - 17 = 0$, the equation at firſt propoſed.

EXAMPLE II.

Let the equation be $x^4 - 6x^3 - 58x^2 - 114x - 11 = 0$, and writing -6, -58, -114, -11 for p, q, r, s, reſpectively, we have $-67 = \alpha$, $-315 = \beta$, and $-1133\tfrac{1}{4} = \zeta$. The numbers β and 2ζ, that is -315 and $-\frac{4533}{2}$, have but one common diviſor 3, that is $n=3$. And the diviſors of $-105=\frac{\beta}{n}$ are 3, 5, 7, 15, 21, 35, and 105. Wherefore I firſt make trial with $3=k$, and dividing $\frac{\beta}{n}$ or -105, by it get the quotient -35, and this ſubtracted from $\frac{1}{2}pk=-3\times3$, leaves 26, whoſe half, 13, ought be equal to l. But $\frac{a+nk^2}{2}$, or $\frac{-67+27}{2}$, that is, -20 is equal to Q; and $Q^2-s=411$, which is indeed diviſible by $n=3$; but the root of the quotient 137 cannot be extracted. Therefore I reject the diviſor 3, and try with $5=k$; by which dividing $\frac{\beta}{n}=-105$, the quotient is -21, and this taken from $\frac{1}{2}pk=-3\times5$, leaves $6=2l$. At the ſame time, $Q\left(=\frac{a+nk^2}{2}\right)=-\frac{75+67}{2}=4$. And Q^2-s, or $16+11$, is diviſible by n, and the root of the quotient 9, that is, 3, coincides with l. Whence I conclude that putting $l=3$, $k=5$, $Q=4$, $n=3$, adding to both ſides of the equation the quantity $nk^2x^2+2nklx+nl^2$, that is, $75x^2+90x+27$, and extracting the roots, it will be

$$x^2+\tfrac{1}{2}px+Q=\sqrt{n}\times\overline{kx+l},\text{ or}$$
$$x^2-3x+4=\pm\sqrt{3}\times\overline{5x+3}.$$

EXAMPLE. III.

In like manner in the equation $x^4-9x^3+15x^2-27x+9=0$ writing $-9,+15,-27,+9$, for p, q, r, s, there reſult $\alpha = -5\frac{1}{4}$, $-50\frac{5}{8} = \beta$, $2\frac{7}{64} = \zeta$. The common diviſors of β and 2ζ, that is, of $\frac{405}{8}$ and $\frac{135}{32}$ are 3, 5, 9, 15, 27, 45, 135; but 9 is a ſquare, and 3, 15, 27, 135 divided by 4 do not leave unity for a remainder, as is required when p is an odd number. Setting theſe aſide there remain only 5 and 45 to be tried for n. Firſt let $n = 5$, and the halves of the odd diviſors of $\frac{\beta}{n} = -\frac{81}{8}$, that is, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$, $\frac{27}{2}$, $\frac{81}{2}$, are to be tried for k. If $k = \frac{1}{2}$, the quotient $-\frac{81}{4}$ of $\frac{\beta}{n}$ divided by k, taken from $\frac{1}{2}pk$ or $-\frac{9}{4}$, leaves $18 = 2l$: and $\mathcal{Q}\left(= \frac{\alpha + nk^2}{2}\right) = -2$, $\mathcal{Q}^2 - s = -5$, which is diviſible by 5, but the root of the quotient -1, which ſhould be $l = 9$, is imaginary. Put next $k = \frac{3}{2}$, and the quotient of $\frac{\beta}{n}$ divided by k, or of $-\frac{81}{8}$ by $\frac{3}{2}$, is $-\frac{27}{4}$. This ſubtracted from $\frac{1}{2}pk = -\frac{27}{4}$, leaves nothing, that is $l = 0$. Again, $\mathcal{Q}\left(= \frac{\alpha + nk^2}{2}\right) = 3$, and $\mathcal{Q}^2 - s = 0$, and $l\left(= \sqrt{\frac{\mathcal{Q}^2 - s}{n}}\right) = 0$. From which coincidence I infer that $n = 5$, $k = \frac{3}{2}$, $l = 0$, and adding $nk^2x^2 + 2nlkx + nl^2$, that is, $\frac{45}{4}x^2$ to both ſides of the equation, I find $x^2 - 4\frac{1}{2}x + 3 = \sqrt{5} \times \frac{3}{2}x$.

Literal equations may be treated much in the ſame way. And, if you put $n = 1$, the ſame Rule will give you the *rational* diviſor of a biquadratic equation, if it admits of one. Thus for the equation $x^4 - x^3 + 5x^2 + 12x - 6 = 0$, putting $n = 1$ I find $k = \frac{5}{2}$, $l = -\frac{5}{2}$, and the equation is reduced to the two quadratics $x^2 - 3x + 3 = 0$ and $x^2 + 2x - 2 = 0$.

When the diviſors of $\frac{\beta}{n}$ are ſo many that it would be troubleſome to make trial with them all for k, their number may be reduced by finding all the diviſors of $\alpha s - \frac{1}{4}r^2$. For to one of theſe, or to its half when odd, the number Q muſt be equal.

The ground of this Rule is as follows.

If a biquadratic equation $x^4 + px^3 + qx^2 + rx + s = 0$, in which p, q, r, s, are the given coefficients with their ſigns, and the equation is ſuppoſed clear of fractions and ſurds; if this equation can be compleated into a ſquare, in the manner already deſcribed, we ſhall have $x^4 + px^3 + qx^2 + rx + s + nk^2x^2 + 2nklx + nl^2 = nk^2x^2 + 2nklx + nl^2 = \overline{x^2 + \frac{1}{2}px + Q}|^2$, that is, $x^4 + px^3 + \overline{q + nk^2} \times x^3 + \overline{r + 2nkl} \times x + s + nl = x^4 + px^3 + \overline{2Q + \frac{1}{4}p^2} \times x^2 + pQ \times x + Q^2$. And comparing the terms, we get theſe *three* equations,

$$1.\ q + nk^2 = 2Q + \tfrac{1}{4}p^2,$$

2.

$$2.\ r + 2nkl = p\mathcal{Q},$$
$$3.\ s + nl^2 = \mathcal{Q}^2;$$

in which there being four unknown quantities, they can be found only by trial.

The values of $\mathcal{Q}$, taken from the firſt and ſecond equations and made equal to each other, give $n = \frac{\frac{1}{2}pq - \frac{1}{8}p^3 - r}{2kl - \frac{1}{2}pk^2}$ (writing, as in the Rule, $q - \frac{1}{4}p^2 = \alpha$, and $r - \frac{1}{2}\alpha p = \beta$) $= \frac{\beta}{k \times \overline{\frac{1}{2}pk - 2l}}$. Whence, if the quantities n, k, l, $\mathcal{Q}$, are to be found, it follows, (1°.) That n being a diviſor of β, giving the quote $k \times \overline{\frac{1}{2}pk - 2l}$, k will be a diviſor of $\frac{\beta}{n}$, giving the quote $\frac{1}{2}pk - 2l$; and that ſubtracting this quote from $\frac{1}{2}pk$, l will be equal to half the remainder. (2°.) In the firſt equation we had $\mathcal{Q} = \frac{\alpha + nk^2}{2}$, and, from the third, $l^2 = \frac{\mathcal{Q}^2 - s}{n}$. (3°.) Becauſe $\mathcal{Q} = \frac{1}{2}\alpha + \frac{1}{2}nk^2$ and $nl^2 = \mathcal{Q}^2 - s$, $n = \frac{\frac{1}{4}\alpha^2 - s}{l^2 - \frac{1}{2}ak^2 - \frac{1}{4}nk^4} = \frac{2s - \frac{1}{2}\alpha^2}{k^2 \times \overline{\alpha + \frac{1}{2}nk^2} - 2l^2}$ (if $\zeta = s - \frac{1}{4}\alpha^2$) $= \frac{2\zeta}{k^2 \times \overline{\alpha + \frac{1}{2}nk^2} - 2l^2}$, that is, n divides 2ζ by $k^2 \times \overline{\alpha + \frac{1}{2}nk^2} - l^2$. And if the ſeveral values of the quantities n, k, l, $\mathcal{Q}$, anſwer to thoſe conditions, or coincide, it is a proof that they have been rightly aſſumed; and that adding to the given equation the quantity $n \times \overline{kx + l}|^2$, it will be compleated into the ſquare $\overline{x^2 + \frac{1}{2}p + \mathcal{Q}}|^2$.

It was ſaid that $\mathcal{Q}$ will always be ſome diviſor of $\alpha s - \frac{1}{4}r^2$. For $\alpha s = \alpha\mathcal{Q}^2 - \alpha n l^2$, and taking from both $\frac{1}{4}r^2 = \frac{1}{4}p^2\mathcal{Q}^2 - p\mathcal{Q}nkl + n^2k^2l^2$, ſeeing the remainder $\alpha\mathcal{Q}^2 - \overline{\alpha + nk^2} \times nl^2 - \frac{1}{4}p^2\mathcal{Q}^2 + p\mathcal{Q}nkl = \alpha\mathcal{Q}^2 - 2\mathcal{Q} \times nl^2 - \frac{1}{4}p^2\mathcal{Q}^2 + p\mathcal{Q}nkl$, has $\mathcal{Q}$ in every term; the thing is manifeſt.

It is needleſs to be particular as to the ſeveral limitations in the Rule, ſeeing they follow eaſily from the algebraical expreſſions of the quantities. You are not, for inſtance, if you ſeek a ſurd diviſor, to take n a ſquare number, for if n is a ſquare number, $\sqrt{n} \times \overline{kx + l}$ would be rational. Or if n is a multiple of a ſquare, as $v \times m^2$, then, at leaſt, $m \times \overline{kx + l}$ would be rational, and n would be depreſſed to v.

Let us examine one caſe, *when p is even and r odd*; and by the Rule n muſt be an odd number, a multiple of 4 more unity.

1. Seeing $\beta = r - \frac{1}{2}\alpha p$, or $\beta + \frac{1}{2}\alpha p = r$, of the numbers β and $\frac{1}{2}\alpha p$ one muſt be even and the other odd, that their ſum r may be odd. If β is odd, its diviſor n muſt be odd likewiſe. Suppoſe β to be even, then $\frac{1}{2}\alpha p$, and conſequently $\frac{1}{2}p$ and α are both odd. But if α is odd, $2\zeta = 2s - \frac{1}{2}\alpha^2$ will be half an odd number, and n its diviſor is odd.

In this caſe, $\mathcal{Q}$ is half an odd number. For let it be an integer, $p\mathcal{Q}$ will be an even number. But if $\mathcal{Q}$ is an integer, ſo muſt l, becauſe

$s + nl^2 = Q^2$; and $2nkl$ muſt be even. And $r + 2nkl$ (an odd number) $= pQ$ an even number, *which is abſurd.*

2. Let N repreſent any number in general, I an odd number; then I ſay, "every odd number is a multiple of *four*, more or leſs *unity*," that is, $I = 4N \pm 1$. "The ſquare of an odd number is $4N + 1$," (that is ſome multiple of 4, more unity;) and "if from ſuch a ſquare there be taken any multiple of 4, the remainder, if greater than unity, will be $4N+1$."

Hence it follows that $n = 4N+1$. For ſeeing $nl^2 = Q^2 - s$; becauſe l and Q are the halves of odd numbers, we have, according to the preſent notation, $\frac{n \times I^2}{4} = \frac{I^2 - 4s}{4}$, or without the common denominator $n \times I^2 = I^2 - 4s$, that is, $n \times \overline{4N + 1} = 4N + 1$, and conſequently, $n = 4N + 1$. For it is not $4N - 1$ but $4N+1$ that can give the product $4N+1$. *

In

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* In the former Editions, there were here inſerted two Rules for the Caſe of $\beta = 0$: which, tho' true, Mr. *Thomas Simpſon* has, in his *Miſcellaneous Tracts*, publiſhed 1757, ſhewn to be unneceſſary. In *this*, therefore, they are omitted.

It is only to be regretted that Mr. *Simpſon* ſhould, through inattention, have placed this inaccuracy, not to the account of the Editor, as he ought to have done, but to that of Mr. MACLAURIN. The whole explanation of Sir *Iſaac*'s Method of Reducing Equations by means of *Surd Diviſors*, is (pag. 213.) profeſſedly a *Supplement*; as is likewiſe the Addition to *Chap.* 14. *Part.* I. And the Edi-

In like manner the other limitations may be determined: and what has been ſaid may lead to the invention or demonſtration of ſimilar Rules for the higher equations of even dimenſions, if any one pleaſes to take the trouble.

CHAP. VIII.

Of the Reſolution of Equations by *Cardan*'s Rule, and others of that kind.

§ 75. WE now proceed to ſhew how an expreſſion of the root of an equation can be obtained that ſhall involve only known quantities. In *Chapter* 11. *Part.* I. we ſhewed how to reſolve *ſimple* equations; and in *Chapter* 13. we ſhewed how to reſolve any *quadratic* equation, by adding to the ſide of the equation that involves the unknown quantity, what was neceſſary to make it a compleat ſquare, and then extracting the ſquare root on both ſides. In § 27 of this *Part*, we gave another method of

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Editor thought he had, in his Preface, ſufficiently intimated that a few ſuch inſertions had been made, and the reaſon why: tho' he cannot recollect any others worth mentioning; if it is not §§ 123, 124, of *Part* II.

of reſolving *quadratic* equations, by taking away the ſecond term: where it appeared that if $x^2-px+q=0$, $x=\frac{1}{2}p\pm\sqrt{\frac{1}{4}p^2-q}$.

§ 76. The ſecond term can be taken away out of any *cubic* equation, by § 25; ſo that they all may be reduced to this form, $x^3 {*} + qx + r = 0$.

Let us ſuppoſe that $x=a+b$; and $x^3+qx+r = a^3+3a^2b+3ab^2+b^3+qx+r = a^3+3ab \times \overline{a+b}+b^3+qx+r = a^3+3abx+b^3+qx+r$ (by ſuppoſing $3ab=-q$) $=a^3+b^3+r=0$.

But $b=-\frac{q}{3a}$, and $b^3=-\frac{q^3}{27a^3}$, and conſequently, $a^3-\frac{q^3}{27a^3}+r=0$; or, $a^6+ra^3+\frac{q^3}{27}$.

Suppoſe $a^3=z$, and you have $z^2+rz=\frac{q^3}{27}$; which is a quadratic whoſe reſolution gives

$$z=-\tfrac{1}{2}r\pm\sqrt{\tfrac{1}{4}r^2+\frac{q^3}{27}}=a^3,$$

and $a=\sqrt[3]{-\frac{1}{2}r\pm\sqrt[2]{\frac{1}{4}r^2+\frac{q^3}{27}}}$; and

$$x=a+b=a-\frac{q}{3a}=\sqrt[3]{-\tfrac{1}{2}r\pm\sqrt[2]{\tfrac{1}{4}r^2+\frac{q^3}{27}}}-\frac{q}{3\times\sqrt[3]{-\tfrac{1}{2}r\pm\sqrt[2]{\tfrac{1}{4}r+\frac{q^3}{27}}}}$$

: in which expreſſions there are only known quantities.

§ 77. The values of x may be found a little differently, thus:

Since $a^3 = -\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$, it follows, that $a^3 + r = +\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$, and $b^3\ (= -a^3 - r) = -\frac{1}{2}r \mp \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$; ſo that $b = \sqrt[3]{-\frac{1}{2}r \mp \sqrt[2]{\frac{1}{4}r^2 + \frac{q^3}{27}}}$; and $x\ (= a + b) =$

$$\sqrt[3]{-\frac{1}{2}r \pm \sqrt[2]{\frac{1}{4}r^2 + \frac{q^3}{27}}} + \sqrt[3]{-\frac{1}{2}r \mp \sqrt[2]{\frac{1}{4}r^2 + \frac{q^3}{27}}};$$

which gives but one value of x, becauſe when, in the value of a the ſurd $\sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$ is poſitive, it is negative in the value of b, and there is only the difference of this ſign in their values. So that we may conclude

$$x = \sqrt[3]{-\frac{1}{2}r + \sqrt[2]{\frac{1}{4}r^2 + \frac{q^3}{27}}} + \sqrt[3]{-\frac{1}{2}r - \sqrt[2]{\frac{1}{4}r^2 + \frac{q^3}{27}}}.$$

§ 79. * The values of x may be diſcovered without exterminating the ſecond term.

Any cubic equation may be reduced to this form,

$$\left.\begin{array}{llll} x^3 - 3px^2 & - 3px & - 2r \\ & + 3p^2x & - p^3 \\ & & + 3pq \end{array}\right\} = 0,$$

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[*] Vid. Phil. Tranſ. 309.

which, by ſuppoſing $x = z + p$, will be reduced to $z^3 * - 3qz - 2r = 0$, in which the ſecond term is wanting. But by the laſt article, ſince $x^3 * - 3qz - 2r = 0$, it follows that $z = \sqrt[3]{r + \sqrt{r^2 - q^3}} + \sqrt[3]{r - \sqrt{r^2 - q^3}}$ (if you ſuppoſe that the cubic root of the binomial $r + \sqrt{r^2 - q^3}$ is $m + \sqrt{n}$) $= m + \sqrt{n} + m - \sqrt{n} = 2m$. And ſince $x = z + p$, it follows that $x = p + 2m$.

§ 79. But as the ſquare root of any quantity is *twofold*, "the cube root is *threefold*." and can be expreſſed three different ways.

Suppoſe the cube root of unit is required, and let $y^3 = 1$, or $y^3 - 1 = 0$, then ſince unit itſelf is a cube root of 1, one of the values of y is 1, ſo that the equation $y - 1 = 0$ ſhall divide the firſt equation $y^3 - 1 = 0$, and the quotient $y^2 + y + 1 = 0$ reſolved, gives $y = \frac{-1 \mp \sqrt{-3}}{2}$; ſo that the three expreſſions of $\sqrt[3]{1}$ are, $\frac{-1 + \sqrt{-3}}{2}$, and $\frac{-1 - \sqrt{-3}}{2}$. And, in general, the cube root of any quantity A^3 may be A, or $\frac{-1 + \sqrt{-3}}{2} \times A$, or $\frac{-1 - \sqrt{-3}}{2} \times A$; ſo that the cube root of the binomial $r + \sqrt{r^2 - q^3}$ may be $m + \sqrt[2]{n}$, as we ſuppoſed above, or $\frac{-1 + \sqrt{-3}}{2}$

$\times m$

$\times\overline{m+\sqrt[2]{n}}$, or $\frac{-1-\sqrt{-3}}{2}\times\overline{m+\sqrt[2]{n}}$. And hence we have three expreſſions for x, *viz.*

$$1^\circ.\ x=p\times 2m,$$
$$2^\circ.\ x=p-m+\sqrt{-3n},$$
$$3^\circ.\ x=p-m-\sqrt{-3n};$$

and theſe give the three roots of the propoſed cubic equation.

EXAMPLE I.

§ 80. Let it be required to find the roots of the equation $x^3-12x^2+41x-42=0$.

Comparing the coefficients of this equation with thoſe of the general equation

$$\left.\begin{array}{l} x^3-3px^2\left.\begin{array}{r}-3q\\+3p^2\end{array}\right\}x\begin{array}{r}-2r\\-p^3\\+3pq\end{array}\end{array}\right\}=0,$$ you find,

$1^\circ.\ 3p=12$, ſo that . . . . $p=4$,

$2^\circ.\ 3p^2-3q\,(=48-3q)=41$ . . . $q=\frac{7}{3}$,

$3^\circ.\ 3pq-p^3-2r(=-36-2r)=-42$ . . $r=3$;

and conſequently, $r^2-q^3=9-\frac{343}{27}=-\frac{100}{27}$, and $r+\sqrt{r^2-q^3}=3+\sqrt{-\frac{100}{27}}$. Now the cube root of this binomial is found to be $-1+\sqrt[2]{-\frac{4}{3}}\,(=m+\sqrt{n})$*. Whence,

$1^\circ.\ x=p+2m=4-2=2.$

$2^\circ.\ x=p-m-\sqrt{-3n}=4+1-\sqrt{4}=5-2=3.$

$3^\circ.\ x=p-m+\sqrt{-3n}=5+2=7.$

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* Section 131. Part I.

So that the three roots of the propoſed equation are 2, 3, 7.

You may find other two expreſſions of the cube root of $3+\sqrt{-\frac{100}{27}}$, beſides $-1+\sqrt{-\frac{4}{3}}$, *viz.* $\frac{3}{2}+\sqrt{-\frac{1}{12}}$, and $-\frac{1}{2}-\sqrt{-\frac{25}{12}}$; but theſe ſubſtituted for $m+\sqrt{n}$ give the ſame values for x, as are already found.

EXAMPLE II.

In the equation $x^3+15x^2+84x-100=0$, you find $p=-5$, $q=-3$, $r=135$, and $r+\sqrt{r^2-q^3}=135+\sqrt{18252}$, whoſe cube root is $3+\sqrt{12}$; ſo that $x\,(=p+2m)=-5+6=1$. The other two values of x, viz. $-8+\sqrt{-36}$, $-8-\sqrt{-36}$, are impoſſible.

After the ſame manner, you will find that the roots of the equation $x^3+x^2-166x+660=0$, are $-15, 7\pm\sqrt{5}$. The Rule by which we may diſcover if any of the roots of an equation are *impoſſible*, ſhall be demonſtrated afterwards.

§ 82. The roots of biquadratic equations may be found by reducing them to cubes, thus.

Let the ſecond term be taken away by the Rule given in *Chap.* 3. And let the equation that reſults be

$$x^{4}*+qx^2+rx+s=0.$$

And let us ſuppoſe this biquadratic to be the product of theſe two quadratic equations,

$$\begin{array}{l} x^2 + ex + f = 0 \\ x^2 - ex + g = 0 \\ \hline \end{array}$$

$$x^{4} * \left.\begin{matrix} +f \\ +g \\ -e^2 \end{matrix}\right\} \times x^2 \left.\begin{matrix} +eg \\ -ef \end{matrix}\right\} \times x + fg = 0.$$

Where e is the coefficient of x in both equations but affected with contrary ſigns; becauſe when the ſecond term is wanting in an equation, the ſum of the affirmative roots muſt be equal to the ſum of the negative.

Compare now the propoſed equation with the above product, and the reſpective terms put equal to each other will give $f + g - e^2 = q$, $eg - ef = r$, $fg = s$. Whence it follows, that $f + g = q + e^2$, and $g - f = \frac{r}{e}$; and conſequently $f + g + g - f \ (= 2g) = q + e^2 + \frac{r}{e}$, and $g = \frac{q + e^2 + \frac{r}{e}}{2}$, the ſame way, you will find, by ſubtraction, &c. $f = \frac{q + e^2 - \frac{r}{e}}{2}$, and $f \times g \ (= s) = \frac{1}{4} \times \overline{q^2 + 2qe^2 + e^4 - \frac{r^2}{e^2}}$; and multiplying by $4e^2$, and ranging the terms, you have this equation,

$$e^6 + 2qe^4 + \overline{q^2 - 4s} \times e^2 - r^2 = 0.$$

Suppoſe $e^2 = y$, and it becomes $y^3 + 2qy^2 +$

q^2

$\overline{q^2-4sy}-r^2=0$, a cubic equation whoſe roots are to be diſcovered by the preceding articles. Then the values of y being found, their ſquare roots will give e (ſince $y=e^2$); and having e, you will find f and g from the equations $f=\dfrac{q+e^2-\frac{r}{e}}{2}$, $g=\dfrac{q+e^2+\frac{r}{e}}{2}$. Laſtly, extracting the roots of the equations $x^2+ex+f=0$, $x^2-ex+g=0$, you will find the four roots of the biquadratic $x^{4}*qx^2+rx+s=0$; for either $x=-\frac{1}{2}e\pm\sqrt{\frac{1}{4}e^2-f}$, or, $x=+\frac{1}{2}e\pm\sqrt{\frac{1}{4}e^2-g}$.

§ 83. Or if you want to find the roots of the biquadratic without taking away the ſecond term; ſuppoſe it to be of this form,

$$x^4-4px^3\left.\begin{matrix}-2q\\+4p^2\end{matrix}\right\}x^2\left.\begin{matrix}-8r\\+4pq\end{matrix}\right\}x\left.\begin{matrix}-4s\\+q^2\end{matrix}\right\}=0,$$

and the values of x will be

$$\left.\begin{matrix}x=p-a\pm\sqrt{p^2+q-a^2-\frac{2r}{a}}\\ \text{and } x=p+a\pm\sqrt{p^2+q-a^2+\frac{2r}{a}}\end{matrix}\right\}, \text{ where}$$

a^2 is equal to the root of the cubic,

$$y^3\left.\begin{matrix}-p^2\\-q\end{matrix}\right\}y^2\left.\begin{matrix}+2pr\\+s\end{matrix}\right\}y-r^2=0.$$

The demonſtration is reduced from the laſt article, as the 78th is from the preceding.

CHAP. IX.

Of the Methods by which you may approximate to the roots of *numeral* Equations by their limits.

§ 84. WHEN any equation is propoſed to be reſolved, firſt find the limits of the roots (by *Chap.* 5.) as for example, if the roots of the equation $x^2-16x+55=0$ are required, you find the limits are 0, 8, and 17, by § 48; that is, the leaſt root is between 0 and 8, and the greateſt between 8 and 17.

In order to find the firſt of the roots, I conſider that if I ſubſtitute 0 for x in $x^2-16x+55$, the reſult is poſitive, *viz.* $+55$, and conſequently any number betwixt 0 and 8 that gives a poſitive reſult, muſt be leſs than the leaſt root, and any number that gives a negative reſult, muſt be greater. Since 0 and 8 are the limits, I try 4, that is, the mean betwixt them, and ſuppoſing $x=4$, $x^2-16x+55=16-64+55=7$, from which I conclude that the root is greater than 4. So that now we have the root limited between 4 and 8. Therefore I next try 6, and ſubſtituting it for x we find $x^2-16x+55=36-96+55=-5$; which reſult being negative, I conclude that 6 is greater than the root required, which therefore is limited now between 4 and 6. And ſubſtituting 5, the mean between them in place of x, I find $x^2 - 16x + 55 = 25 - 80 + 55 = 0$; and conſequently 5 is the leaſt root of the equation. After the ſame manner you will diſcover 11 to be the greateſt root of that equation.

§ 85. Thus by diminiſhing the greater, or increaſing the leſſer limit, you may diſcover the true root when it is a commenſurable quantity. But by proceeding after this manner, when you have two limits, the one greater than the root, the other leſſer, that differ from one another but by unit, then you may conclude the root is *incommenſurable.*

We may however, by continuing the operation in fractions, approximate to it. As if the equation propoſed is $x^2 - 6x + 7 = 0$, if we ſuppoſe $x = 2$, the reſult is $4 - 12 + 7 = -1$, which being negative, and the ſuppoſition of $x = 0$ giving a poſitive reſult, it follows that the root is betwixt 0 and 2. Next we ſuppoſe $x = 1$; whence $x^2 - 6x + 7 = 1 - 6 + 7 = +2$, which being poſitive, we infer the root is betwixt 1 and 2, and conſequently incommenſurable. In order to approximate to it, we ſuppoſe $x = 1\frac{1}{2}$, and find $x^2 - 6x + 7 = 2\frac{1}{4} - 9 + 7 = \frac{1}{4}$; and this reſult being poſitive, we infer the root muſt be betwixt 2 and $1\frac{1}{2}$. And therefore we try $1\frac{3}{4}$, and find $x^2 - 6x + 7 = \frac{49}{16} - \frac{42}{4} + 7 = 3\frac{1}{16} - 10\frac{8}{16} + 7 = -\frac{7}{16}$, which is negative; ſo that we conclude the root to be betwixt $1\frac{3}{4}$ and $1\frac{1}{2}$. And therefore we try next $1\frac{5}{8}$, which giving alſo a negative reſult, we conclude the root is betwixt $1\frac{1}{2}$ (or $1\frac{4}{8}$) and $1\frac{5}{8}$. We try therefore $1\frac{9}{16}$, and the reſult being poſitive, we conclude that the root muſt be betwixt $1\frac{9}{16}$ and $1\frac{10}{16}$, and therefore is nearly $1\frac{19}{32}$.

§ 86. Or you may approximate more eaſily by transforming the equation propoſed into another whoſe roots ſhall be equal to 10, 100, or 1000 times the roots of the former (by § 29.) and taking the limits greater in the ſame proportion. This transformation is eaſy; for you are only to multiply the ſecond term by 10, 100, or 1000, the third term by their ſquares, the fourth by their cubes, *&c.* The equation of the laſt example is thus transformed into $x^2 - 600x + 70000 = 0$, whoſe roots are 100 times the roots of the propoſed equation, and whoſe limits are 100 and 200. Proceeding as before, we try 150, and find $x^2 - 600x + 70000 = 22500 - 90000 + 70000 = 2500$, ſo that 150 is leſs than the root. You next try 175, which giving a negative reſult muſt be greater than the root: and thus proceeding you find the root to be betwixt 158 and 159: from which you infer that the leaſt root of the propoſed equation $x^2 - 6x + 7 = 0$ is betwixt 1.58 and 1.59, being the hundreth part of the root of $x^2 - 600x + 70000 = 0$.

§ 87. If the cubic equation $x^3 - 15x^2 + 63x - 50 = 0$ is propoſed to be reſolved, the equation of the limits will be (by § 48) $3x^2 - 30x + 63 = 0$, or $x^2 - 10x + 21 = 0$, whoſe roots are 3, 7; and by ſubſtituting 0 for x the value of $x^3 - 15x^2 + 63x - 50$ is negative, and by ſubſtituting 3 for x, that quantity becomes poſitive, $x = 1$ gives it negative, and $x = 2$ gives it poſitive, ſo that the root is between 1 and 2, and therefore incommenſurable. You may proceed as in the foregoing examples to approximate to the root. But there are other methods by which you may do that more eaſily and readily; which we proceed to explain.

§ 88. When you have diſcovered the value of the root to leſs than an unit (as in this example, you know it is a little above 1) ſuppoſe the difference betwixt its real value and the number that you have found nearly equal to it, to be repreſented by f; as in this example. Let $x = 1 + f$. Subſtitute this value for x in this equation, thus,

$$\begin{array}{rcl} x^3 & = & 1 + 3f + 3f^2 + f^3 \\ -15x^2 & = & -15 - 30f - 15f^2 \\ +63x & = & 63 + 63f \\ -50 & = & -50 \\ \hline \end{array}$$

$$x^3 - 15x^2 + 63x - 50 = -1 + 36f - 12f^2 + f^3 = 0.$$

Now becauſe f is ſuppoſed leſs than unit, its powers f^2, f^3, may be neglected in this approximation; ſo that aſſuming only the two firſt terms we have $-1+36f=0$, or, $f=\frac{1}{36}=.027$; ſo that x will be nearly 1.027.

You may have a nearer value of x by conſidering, that ſeeing $-1+36f-12f^2+f^3=0$, it follows that

$$f=\frac{1}{36-12f+f^2}\text{ (by ſubſtituting }\frac{1}{36}\text{ for }f)$$

$$\text{nearly}=\frac{1}{36-12\times\frac{1}{36}+\frac{1}{36}\times\frac{1}{36}}=\frac{1296}{46225}=.02803.$$

§ 89. But the value of f may be corrected and determined more accurately by ſuppoſing g to be the difference betwixt its real value, and that which we laſt found nearly equal to it. So that $f=.02803+g$. Then by ſubſtituting this value for f in the equation $f^3-12f^2+36f-1=0$, it will ſtand as follows,

$$\left.\begin{array}{rl} f^3= & 0.0000220226+0.002357g+0.0849g^2+g^3 \\ -12f^2= & -.00942816\quad -0.67272g\quad -12g^2 \\ +36f= & 1.00908\qquad\quad +36g \\ -1= & -1. \end{array}\right\}=0$$

$$=-0.0003261374+35.329637g-11.9195g^2+g^3=0.$$

Of which the firſt two terms, neglecting the reſt, give $35.329637\times g=0.0003261374$, and $g=\frac{.0003261374}{35.329637}=0.00000923127$. So that $f=0.02803923127$; and $x=1+f=1.02803923127$; which is very near the true root of the equation that was propoſed.

If ſtill a greater degree of exactneſs is required, ſuppoſe h equal to the difference betwixt the true value of g, and that we have already found, and proceeding as above you may correct the value of g.

§ 90. For another example; let the equation to be reſolved be $x^3 - 2x - 5 = 0$, and by ſome of the preceding methods you diſcover one of the roots to be between 2 and 3. Therefore you ſuppoſe $x = 2 + f$, and ſubſtituting this value for it, you find

$$\left.\begin{array}{rl} x^3 = & 8 + 12f + 6f^2 + f^3 \\ -2x = & -4 - 2f \\ -5 = & -5 \end{array}\right\} = 0$$

$$= -1 + 10f + 6f^2 + f^3;$$

from which we find that $10f = 1$ nearly, or $f = 0.1$. Then to correct this value, we ſuppoſe $f = 0.1 + g$, and find

$$\left.\begin{array}{rl} f^3 = & 0.001 + 0.03g + 0.3g^2 + g^3 \\ 6f^2 = & 0.06 + 1.2\,g + 6.\,g^2 \\ 10f = & 1. \quad + 10.\,g \\ -1 = & -1. \end{array}\right\} = 0$$

$$= 0.061 + 11.23g + 6.3g^2 + g^3,$$

ſo that $g = \dfrac{-0.061}{11.23} = -0.0054$.

Then by ſuppoſing $g = -.0054 + h$, you may correct its value, and you will find that the root required is nearly 2.09455147.

§ 91. It is not only one root of an equation that can be obtained by this method, but, by making uſe of the other limits, you may diſcover the other roots in the ſame manner. The equation of § 87, $x^3 - 15x^2 + 63x - 50 = 0$, has for its limits 0, 3, 7, 50. We have already found the leaſt root to be nearly 1.028039. If it is required to find the middle root, you proceed in the ſame manner to determine its neareſt limits to be 6 and 7; for 6 ſubſtituted for x gives a poſitive, and 7 a negative reſult. Therefore you may ſuppoſe $x = 6 + f$, and by ſubſtituting this value for x in that equation, you find $f^3 + 3f^2 - 9f + 4 = 0$, ſo that $f = \frac{4}{9}$ nearly. Or ſince $f = \frac{4}{9 - 3f - f^2}$, it is (by ſubſtituting $\frac{4}{9}$ for f) $f = \frac{4}{9 - \frac{3}{4} - \frac{16}{81}} = \frac{324}{605}$, whence $x = 6 + \frac{324}{605}$ nearly. Which value may ſtill be corrected as in the preceding articles. After the ſame manner you may approximate to the value of the higheſt root of the equation.

§ 92. "In all theſe operations, you will approximate ſooner to the value of the root, if you take the three laſt terms of the equation, and extract the root of the quadratic equation conſiſting of theſe three terms."

Thns, in § 88, inſtead of the two laſt terms of the equation $f^3 - 12f^2 + 36f - 1 = 0$, if you take the three laſt and extract the root of the quadratic $12f^2 - 36f + 1 = 0$, you will find $f = .028031$, which is much nearer the true value than what you diſcover by ſuppoſing $36f - 1 = 0$.

It is obvious that this method extends to all equations.

§ 93. "By aſſuming equations affected with general coefficients, you may, by this method, deduce General Rules or *Theorems* for approximating to the roots of propoſed equations of whatever degree."

Let $f^3 - pf^2 + qf - r = 0$ repreſent the equation by which the fraction f is to be determined, which is to be added to the limit, or ſubtracted from it, in order to have the near value of x. Then $qf - r = 0$ will give $f = \frac{r}{q}$. But ſince $f = \frac{r}{f^2 - pf + q}$, by ſubſtituting $\frac{r}{q}$ for f, we have this Theorem for finding f nearly, *viz.*

$$f = \frac{r}{\frac{r^2}{q^2} - \frac{pr}{q} + q} = \frac{q^2 \times r}{q^3 - pqr + r^2}.$$

After the ſame manner, if it is a biquadratic, by which f is to be determined, as $f^4 - pf^3 + qf^2 - rf + s = 0$, then f being very little, we ſhall have $f = \frac{s}{r}$; which value is corrected by conſidering that $f = \frac{s}{r - qf + pf^2 - f^3}$ (by ſubſtituting $\frac{s}{r}$ for f) $= \frac{s}{r - \frac{qs}{r} + \frac{ps^2}{r^2} - \frac{s^3}{r^3}}$, whence we have this Theorem for all biquadratic equations,

$$f = \frac{r^3 \times s}{-s^3 + ps^2r - qsr^2 + r^4}.$$

§ 94. Other Theorems may be deduced by aſſuming the three terms of the equation, and extracting the root of the quadratic which they form.

Thus, to find the value of f in the equation $f^3 - pf^2 + qf - r = 0$ where f is ſuppoſed to be very little, we neglect the firſt term f^3, and extract the root of the quadratic $pf^2 - qf + r = 0$, or of $f^2 - \frac{q}{p} \times f + \frac{r}{p} = 0$; and we find $f = \frac{q}{2p} \pm \sqrt{-\frac{r}{p} + \frac{q^2}{4p^2}} = \frac{q \pm \sqrt{q^2 - 4pr}}{2p}$ nearly.

But this value of f may be corrected by ſuppoſing it equal to m, and ſubſtituting m^3 for f^3 in the equation $f^3 - pf^2 + qf - r = 0$, which will give $m^3 - pf^2 + qf - r = 0$, and $pf^2 - qf + r - m^3 = 0$; the reſolution of which quadratic equation gives $f = \frac{q \pm \sqrt{q^2 - 4pr + 4pm^3}}{2p}$, very near the true value of f.

After the ſame manner you may find like Theorems for the roots of biquadratic equations, or of equations of any dimenſion whatever.

§ 95. In general, let $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} +$, &c. $+ A = 0$ repreſent an equation of any dimenſions n, where A is ſuppoſed to repreſent the abſolute known term of the equation. Let k repreſent the limit next leſs than any of the roots, and ſuppoſing $x = k + f$, ſubſtitute the powers of $k + f$ inſtead of the powers of x, and there will ariſe $\overline{k+f}^n + p \times \overline{k+f}^{n-1} + q \times \overline{k+f}^{n-2} + r \times \overline{k+f}^{n-3}$, &c. $+ A = 0$, or by involution, diſpoſing the terms according to the dimenſions of f . . . .

$$\left.\begin{array}{l} k^n \quad + nk^{n-1} \times f + n \times \frac{n-1}{2} k^{n-2} f^2 +, \text{\&c.} \\ pk^{n-1} + p \times \overline{n-1} k^{n-2} \times f + p \times \overline{n-1} \times \frac{n-2}{2} k^{n-3} f^2, + \text{\&c.} \\ qk^{n-2} + q \times \overline{n-2} k^{n-3} \times f + q \times \overline{n-2} \times \frac{n-3}{2} k^{n-4} f^2, + \text{\&c.} \\ rk^{n-3} + r \times \overline{n-3} k^{n-4} \times f + r \times \overline{n-3} \times \frac{n-4}{2} k^{n-5} f^2, + \text{\&c.} \\ \qquad \text{\&c.} \end{array}\right\} \begin{array}{l} +A \\ =0, \end{array}$$

where neglecting all the powers of *f* after the firſt two terms, you find

$$f=\frac{-A-k^n-pk^{n-1}-qk^{n-2}-rk^{n-3},\ \&c.}{nk^{n-1}+p\times\overline{n-1}k^{n-2}+q\times\overline{n-2}k^{n-3}+r\times\overline{n-3}k^{n-4},\ \&c.},\ \text{and}$$

$$x(=k+f)=\frac{-A+\overline{n-1}k^n+p\times\overline{n-2}k^{n-1}+q\times\overline{n-3}k^{n-2}+r\times\overline{n-4}k^{n-3},\ \&c.}{nk^{n-1}+p\times\overline{n-1}k^{n-2}+q\times\overline{n-2}k^{n-3}+r\times\overline{n-3}k^{n-4},\ \&c.}$$

whence particular Theorems for extracting the roots of equations may be deduced.

§ 96. " By this method you may diſcover Theorems for approximating to the roots of *pure powers*;" as to find the n root of any number A; ſuppoſe k to be the neareſt leſs root in integers, and that $k+f$ is the true root, then ſhall $k^n + nk^{n-1}f + n \times \frac{n-1}{2}k^{n-2}f^2$, &c. $= A$; and, aſſuming only the two firſt terms, $f = \frac{A-k^n}{nk^{n-1}}$: or, more nearly, taking the three firſt terms,

$$f = \frac{A-k^n}{nk^{n-1} + n \times \frac{n-1}{2}k^{n-2}f}$$, and (taking $\frac{A-k^n}{nk^{n-1}} = f$)

$$f = \frac{A-k_n}{nk^{n-1} + \frac{n^2-n}{2}k^{n-2} \times \frac{A-k_n}{nk^{n-1}}} = \frac{A-k^n}{nk^{n-1} + \frac{n-1}{2k} \times \overline{A-k^n}}$$

(putting $m = A-k^n$) $= \frac{km}{nk^n + \frac{n-1}{2} \times m}$; which is a *rational* Theorem for approximating to f.

You may find an *irrational* Theorem for it, by aſſuming the three firſt terms of the power of $k+f$, viz. $k^n + nk^{n-1}f + n \times \frac{n-1}{2}k^{n-2}f^2 = A$. For $nk^{n-1}f + n \times \frac{n-1}{2}k^{n-2}f^2 = A - k^n = m$; and reſolving this quadratic equation, you find

$$f = -\frac{k}{n-1} \pm \sqrt{\frac{2m}{n \times \overline{n-1} \times k^{n-2} + \frac{k^2}{\overline{n-1}^2}}} =$$

$$-\frac{k}{n-1} \pm \sqrt{\frac{2mn - 2m + nk^n}{n \times \overline{n-1}^2 k^{n-2}}}.$$

In the application of theſe Theorems, when a near value of f is obtained, then adding it to k, ſubſtitute the aggregate in place of k in the formula, and you will by a new operation, obtain a more correct value of the root required; and, by thus proceeding, you may arrive at any degree of exactneſs.

Thus to obtain the cube root of 2, ſuppoſe $k = 1$, and $f \left(= \frac{km}{nk^n + \frac{n-1}{2} m}\right) = \frac{1}{4} = 0.25$. In the ſecond place, ſuppoſe $k = 1.25$, and f will be found, by a new operation, equal to 0.009921, and conſequently, $\sqrt[3]{2} = 1\ 259921$ nearly. By the irrational Theorem, the ſame value is diſcovered for $\sqrt[3]{2}$.

CHAP. X.

Of the Method of *Series* by which you may approximate to the roots of *literal* equations.

§ 97. IF there be only two letters, x and a, in the propoſed equation, ſuppoſe a equal to unit, and find the root of the numeral equation that ariſes from the ſubſtitution, by the rules of the laſt chapter. Multiply theſe roots by a, and the products will give the roots of the propoſed equation.

Thus the roots of the equation $x^2 - 16x + 55 = 0$ are found, in § 84, to be 5 and 11. And therefore the roots of the equation $x^2 - 16ax + 55a^2 = 0$, will be $5a$ and $11a$. The roots of the equation $x^3 + a^2x - 2a^3 = 0$ are found by enquiring what are the roots of the numeral equation $x^3 + x - 2 = 0$, and ſince one of theſe is 1, it follows that one of the roots of the propoſed equation is a; the other two are *imaginary*.

§ 98. If the equation to be reſolved involves more than two letters, as $x^3 + a^2x - 2a^3 + ayx - y^3 = 0$, then the value of x may be exhibited in a ſeries having its terms compoſed of the powers of a and y with their reſpective coefficients;

ficients; which will "*converge the ſooner the leſs* y *is in reſpect of* a, *if the terms are continually multiplied by the powers of* y, *and divided by thoſe of* a." Or, "will *converge the ſooner the greater* y *is in reſpect of* a, *if the terms be continually multiplied by the powers of* a, *and divided by thoſe of* y." Since when y is very little in reſpect of a, the terms $y, \frac{y^2}{a}, \frac{y^3}{a^2}, \frac{y^4}{a^3}, \frac{y^5}{a^4}$, &c. decreaſe very quickly. If y vaniſh in reſpect of a, the ſecond term will vaniſh in reſpect of the firſt, ſince $\frac{y^2}{a} : y :: y : a$. And after the ſame manner $\frac{y^3}{a^2}$ vaniſhes in reſpect of the term immediately preceding it.

But when y is vaſtly great in reſpect of a, then a it vaſtly great in reſpect of $\frac{a^2}{y}$, and $\frac{a^2}{y}$ in reſpect of $\frac{a^3}{y^2}$; ſo that the terms $a, \frac{a^2}{y}, \frac{a^3}{y^2}, \frac{a^4}{y^3}, \frac{a^5}{y^4}$, &c. in this caſe *decreaſe* very ſwiftly. In either caſe, the ſeries converge ſwiftly that conſiſt of ſuch terms; and a few of the firſt terms will give a near value of the root required.

§ 99. If a ſeries for x is required from the propoſed equation that ſhall converge the ſooner, the leſs y is in reſpect of a; to find the firſt term of this ſeries, we ſhall ſuppoſe y to vaniſh; and extracting the root of the equation

$x^3 + a^2x - 2a^3 = 0$, conſiſting of the remaining parts of the equation that do not vaniſh with y, we find, by § 97, that $x = a$; which is the true value of x when y vaniſhes, but is only near its value when y does not vaniſh, but only is very little. To get a value ſtill nearer the true value of x, ſuppoſe the difference of a from the true value to be p, or that $x = a + p$. And ſubſtituting $a + p$ in the given equation for x, you will find,

$$\left.\begin{aligned} x^3 &= a^3 + 3a^2p + 3ap^2 + p^3 \\ +a^2x &= a^3 + a^2p \\ -2a^3 &= -2a^3 \\ +ayx &= a^2y + apy \\ -y^3 &= -y^3 \end{aligned}\right\} = 0$$

$$= \left.\begin{aligned} 4a^2p + 3ap^2 + p^3 \\ a^2y + apy - y^3 \end{aligned}\right\} = 0.$$

But ſince, by ſuppoſition, y and p are very little in reſpect of a, it follows that the terms $4a^2p$, a^2y, where y and p are ſeparately of the *leaſt* dimenſions, are vaſtly great in reſpect of the reſt; ſo that, in determining a near value of p, the reſt may be neglected: and from $4a^2p + a^2y = 0$, we find $p = -\frac{1}{4}y$. So that $x = a + p = a - \frac{1}{4}y$, nearly.

Then to find a nearer value of p, and conſequently of x, ſuppoſe $p = -\frac{1}{4}y + q$, and ſubſtituting this value for it in the laſt equation, you will find,

$$\left.\begin{array}{rl} p^3 = & -\frac{1}{64}y^3+\frac{3}{16}y^2q-\frac{3}{4}yq^2+q^3 \\ 3ap^2 = & \frac{3}{16}ay^2-\frac{3}{2}ayq+3aq^2 \\ 4a^2p = & -a^2y+4a^2q \\ ayp = & -\frac{1}{4}ay^2+ayq \\ a^2y = & a^2y \\ -y^3 = & -y^3 \end{array}\right\}=0$$

$$\left.\begin{array}{l} =-\frac{65}{64}y^3+\frac{3}{16}y^2q-\frac{3}{4}yq^2+q^3 \\ -\frac{1}{16}ay^2-\frac{1}{2}ayq+3aq^2 \\ +4a^2q \end{array}\right\}=0.$$

And ſince, by the ſuppoſition, q is very little in reſpect of p, which is nearly $=-\frac{1}{4}y$, therefore q will be very little in reſpect of y; and conſequently all the terms of the laſt equation will be very little in reſpect of theſe two, *viz.* $-\frac{1}{16}ay^2$, $+4a^2q$, where y and q are of leaſt dimenſions ſeparately: particularly the term $-\frac{1}{2}ayq$ is little in reſpect of $4a^2q$, becauſe y is very little in reſpect of a; and it is little in reſpect of $-\frac{1}{16}ay^2$, becauſe q is little in reſpect of y.

Neglect therefore the other terms and ſuppoſing $-\frac{1}{16}ay^2+4a^2q=0$, you will have $q=\frac{1}{64}\times\frac{y^2}{a}$; ſo that $x=a-\frac{1}{4}y+\frac{1}{64}\times\frac{y^2}{a}$. And by proceeding in the ſame manner you will find $x=a-\frac{y}{4}+\frac{y^2}{64a}-\frac{131y^3}{512a^2}+\frac{509y^4}{16384a^3}-$, &c.

§ 100. When it is required to find a ſeries for x that ſhall converge ſooner, the *greater* y is in reſpect of any quantity a, you need only ſuppoſe a to be very little in reſpect of y, and proceed by the ſame reaſoning as in the laſt example on the ſuppoſition of y being very little.

Thus, to find a value for x in the equation $x^3 - a^2x + ayx - y^3 = 0$ that ſhall converge the ſooner the greater y is in reſpect of a, ſuppoſe a to vaniſh, and the remaining terms will give $x^3 - y^3 = 0$, or $x = y$. So that when y is vaſtly great, it appears that $x = y$ nearly.

But to have the value of x more accurately, put $x = y + p$, then

$$\left.\begin{array}{rl} x^3 = & y^3 + 3y^2p + 3yp^2 + p^3 \\ -a^2x = & -a^2y - a^2p \\ +ayx = & ay^2 + ayp \\ -y^3 = & -y^3 \end{array}\right\} = 0$$

$$= +3y^2p + 3yp^2 + p^3 - a^2y - a^2p \\ + ay^2 + ayp:$$

where the terms $3y^2p + ay^2$ become vaſtly greater than the reſt, y being vaſtly greater than a or p; and conſequently $p = -\frac{1}{3}a$ nearly.

Again, by ſuppoſing $p = -\frac{1}{3}a + q$, you will transform the laſt equation into

$$\left.\begin{array}{l} -\frac{8}{27}a^3 + 3y^2q + 3yq^2 + q^3 \\ -\ a^2y - \ ayq - \ aq^2 \\ \qquad -\frac{2}{3}a^2q \end{array}\right\} = 0;$$

where the two terms $3qy^2 - a^2y$ muſt be vaſtly greater than any of the reſt, a being vaſtly leſs than y, and q vaſtly leſs than a, by the ſuppoſition; ſo that $3qy^2 - a^2y = 0$, and $q = \frac{a^2}{3y}$ nearly. By proceeding in this manner, you may correct the value of y, and find that

$$x = y - \tfrac{1}{3}a + \frac{a^2}{3y} + \frac{a^3}{81y^2} - \frac{8a^4}{243y^3}, \text{ \&c.}$$

which ſeries converges the ſooner the greater y is ſuppoſed to be taken in reſpect of a.

§ 101. In the ſolution of the firſt *Example* thoſe terms were always compared in order to determine p, q, r, &c. in which y and thoſe quantities p, q, r, &c. were *ſeparately* of feweſt dimenſions. But in the ſecond *Example*, thoſe terms were compared in which a and the quantities p, q, r, &c. were of leaſt dimenſions ſeparately. And theſe always are the proper terms to be compared together, becauſe they become vaſtly greater than the reſt, in the reſpective hypotheſes.

In general; to determine the firſt, or any, term in the ſeries, *ſuch terms of the equation are to be aſſumed together only*, as *will be found to become vaſtly greater than the other terms*; that is, which give a value of x, which ſubſtituted for it in all the terms of the equation ſhall raiſe the dimenſions of the other terms all above, or all below, the dimenſions of the aſſumed terms, according as y is ſuppoſed to be vaſtly little, or vaſtly great in reſpect of a.

Thus to determine the firſt term of a converging ſeries expreſſing the value of x in the laſt equation $x^3 - a^2x + ayx - y^3 = 0$, the terms ayx and $-y^3$ are not to be compared together, for they would give $x = \frac{y^2}{a}$, which ſubſtituted for x, the equation becomes

$$\frac{y^6}{a^3} - ay^2 + y^3 - y^3 = 0,$$

where the firſt term is of more dimenſions than the aſſumed terms ayx, $-y^3$; and the ſecond of fewer: ſo that the two firſt terms cannot be neglected in reſpect of the two laſt, neither when y is very great nor very little, compared with a. Nor are the terms x^3, ayx, fit to be compared together in order to obtain the firſt term of a ſeries for x, for the like reaſon.

But x^3 may be compared with $-a^2x$, as alſo $-a^2x$ with $-y^3$ for that end. Theſe two give the firſt term of a ſeries that converges the ſooner the leſs y is; as $x^3 = y^3$ gives the firſt term of a ſeries that converges the ſooner the greater y is. The laſt ſeries was given in the preceding article. The comparing x^3 with $-a^2x$ gives theſe two ſeries,

$$x = a - \tfrac{1}{2}y - \frac{y^2}{8a} + \frac{7y^3}{16a^2} - \frac{59y^4}{128a^3}, \text{ \&c.}$$

$$x = -a + \tfrac{1}{2}y + \frac{y^2}{8a} + \frac{9y^3}{16a^2} + \frac{69y^4}{128a^3}, \text{ \&c.}$$

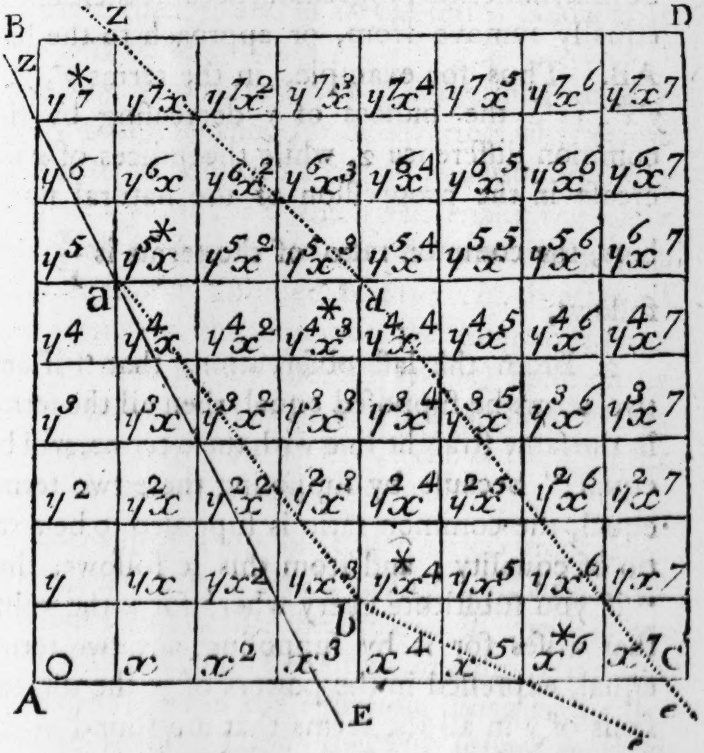
The comparing $-a^2x$ with $-y^3$ gives

$$x = -\frac{y^3}{a^2} - \frac{y^4}{a^3} - \frac{y^5}{a^4} - \frac{y^6}{a^5} -, \text{ \&c.}$$

And theſe ſeries give three values of x when y is very little; the laſt of which is itſelf alſo very little in that caſe, as it appears indeed from the equation, that when y vaniſhes, the three values of x become $+a$, $-a$, and 0, becauſe when y vaniſhes, the equation becomes $x^3-a^2x=0$, whoſe roots are a, $-a$, 0.

§ 102. It appears ſufficiently from what we have ſaid, that when an equation is propoſed involving x and y, and the value of x is required in a converging ſeries, the difficulty of finding the firſt term of the ſeries is reduced to this; "to find what terms aſſumed in order to determine a value of x expreſſed in ſome dimenſions of y and a will give ſuch a value of it, as ſubſtituted for it in the other tems will make them all of more dimenſions of y, or all of leſs dimenſions of y than thoſe aſſumed terms."

To determine this, draw BA and AC at right angles to each other, compleat the parallelogram ABCD and divide it into equal ſquares, as in the figure. In theſe ſquares place the powers of x from A towards C, and the powers of y from A towards B, and in any other ſquare place that power of x that is directly below it in the line AC, and that power of y that is in a parallel with it in the line AB; ſo that the index of x in any ſquare may expreſs its diſtance from the line AB, and the index of y in any ſquare may expreſs its diſtance from the line AC. Of this ſquare we are to obſerve,



* y^7	y^7x	y^7x^2	y^7x^3	y^7x^4	y^7x^5	y^7x^6	y^7x^7
y^6	y^6x	y^6x^2	y^6x^3	y^6x^4	y^6x^5	y^6x^6	y^6x^7
y^5	* y^5x	y^5x^2	y^5x^3	y^5x^4	y^5x^5	y^5x^6	y^6x^7
y^4	y^4x	y^4x^2	* y^4x^3	y^4x^4	y^4x^5	y^4x^6	y^4x^7
y^3	y^3x	y^3x^2	y^3x^3	y^3x^4	y^3x^5	y^3x^6	y^3x^7
y^2	y^2x	y^2x^2	y^2x^3	y^2x^4	y^2x^5	y^2x^6	y^2x^7
y	yx	yx^2	yx^3	* yx^4	yx^5	yx^6	yx^7
O	x	x^2	x^3	x^4	x^5	* x^6	x^7

1. That the terms are not only in geometrical progreſſion in the vertical column AB, or the horizontal AC, and their parallels; but alſo in the terms taken in any oblique ſtraight line whatever; for in any ſuch terms it is manifeſt that the *indices* of *y* and *x* will be in arithmetical progreſſion. The indices of *y*, becauſe thoſe terms will remove equally from the line AC, or approach equally to it, and the indices

of y in any ſuch terms are as their diſtances from that line AC. The indices of x will alſo be in arithmetical progreſſion, becauſe theſe terms equally remove from, or approach to the line AB. Thus for example, in the terms y^7, y^5x, y^3x^2, yx^3, the indices of y decreaſing by the common difference 2, while the indices of x increaſe in the progreſſion of the natural numbers, the common ratio of the terms is $\frac{x}{y^2}$. It follows,

2. From the laſt obſervation, that "if any two terms be ſuppoſed equal, then all the terms in the ſame ſtraight line with theſe terms, will be equal;" becauſe by ſuppoſing theſe two terms equal, the common ratio is ſuppoſed to be a ratio of equality; and from this it follows, that "if you ſubſtitute every where for x the value that ariſes for it by ſuppoſing any two terms equal, expreſſed in the powers of y, the dimenſions of y in all the terms that are found in the ſame ſtraight line will be equal"; but "the dimenſions of y in the terms *above* that line will be *greater* than in thoſe in that line;" and "the dimenſions of y in the terms *below* the ſaid line will be *leſs* than its dimenſions in that line." Thus, by ſuppoſing $y^7=yx^3$, we find $x^3=y^6$, or $x=y^2$; and ſubſtituting this value for x in all the ſquares, the dimenſions of y in the terms y^7, y^5x, y^3x^2, yx^3, which are all found in

the ſame ſtraight line, will be 7, but the dimenſions in all the terms above that line will be more than 7, and in all the terms below that line will be leſs than 7.

§ 103. From theſe two obſervations we may eaſily find a method for diſcovering what terms ought to be aſſumed from an equation in order to give a value for x which ſhall make the other terms all of *higher*, or all of *lower* dimenſions of y than the aſſumed terms: *viz.* "after all the terms of the equation are ranged in their proper ſquares (by the laſt article) ſuch terms are to be aſſumed as lie in a ſtraight line, ſo that the other terms either lie all above the ſtraight line, or fall all below it."

For example, ſuppoſe the equation propoſed is $y^7 - ay^5x + y^4x^3 + a^2yx^4 - ax^6 = 0$, then marking with an aſteriſk the ſquares in the laſt article which contain the ſame dimenſions of x and y as the terms in the equation, imagine a ruler ZE to revolve about the firſt ſquare marked at y^7, and as it moves from A towards C, it will firſt meet the term ay^5x, and while the ruler joins theſe two terms, all the other terms lie above it: from which you infer, that by ſuppoſing theſe terms equal, you ſhall obtain a value of x, which ſubſtituted for it, will give all the other terms of higher dimenſions of y, than thoſe terms: and hence we conclude that the value of x deduced from ſuppoſing theſe terms equal, *viz.* $\frac{y^2}{a}$, is the firſt term of a ſeries that will converge the ſooner the leſs y is in reſpect of a.

If the ruler be made to revolve about the ſame ſquare the contrary way from D towards C, it will firſt meet the term y^4x^3, and by ſuppoſing $y^7 + y^4x^3 = 0$, we find $y = x$, which gives the firſt term of a ſeries for x, that converges the ſooner the greater that y is. And this is the celebrated Rule invented by Sir *Iſaac Newton* for this purpoſe.

§ 104. This Rule may be extended to equations having terms that involve powers of x and y with *fractional* or *ſurd* indices; "by taking diſtances from A in the lines AC and AB proportional to theſe fractions and ſurds," and thence determining the ſituation of the terms of the propoſed equation in the parallelogram ABCD.

It is to be obſerved alſo, that when the line joining any two terms has all the other terms on one ſide of it, by them you may find the firſt term of a converging ſeries for x, and thus "various ſuch ſeries can be deduced from the ſame equation." As, in the laſt Example, the line joining y^5x and yx^4 has all the terms above it; and therefore ſuppoſing $-ay^5x + a^2yx^4 = 0$, we find $x^3 = \frac{y^4}{a}$, and $x = \frac{y^{\frac{4}{3}}}{a^{\frac{1}{3}}}$, which is the firſt term of another converging ſeries for x. Again, the ſtraight line joining yx^4 and x^6 has all the other terms above it, and therefore, ſuppoſing $a^2yx^4 - ax^6 = 0$, we find $ay = x^2$, and $x = a^{\frac{1}{2}}y^{\frac{1}{2}}$, the firſt term of another ſeries for x, converging alſo the ſooner the leſs y is. There are two ſeries converging the ſooner the greater y is, to be deduced from ſuppoſing $y^7 = -y^4x^3$, or $y^4x^3 = ax^6$. And, to find all theſe ſeries, "deſcribe a *polygon Zabed*, having a term of the equation in each of its angles, and including all the other terms within it, then a ſeries may be found for x, by ſuppoſing any two terms equal that are placed in any two adjacent angles of the polygon."

§ 105. If the ruler ZE be made to move parallel to itſelf, all the terms which it will touch at once will be of the ſame dimenſions of y: for they will bear the ſame proportion to one another as the terms in the line ZE themſelves. The terms which the Ruler will touch firſt will have fewer dimenſions of y, than thoſe it touches afterwards in the progreſs of its motion, if it moves towards D; but more dimenſions than they, if it moves towards A. The terms in the ſtraight line ZE, ſerve to determine the firſt term of the converging ſeries required. Theſe with the terms it touches afterwards ſerve to determine the ſucceeding terms of the converging ſeries; all the reſt vaniſhing compared with theſe, when y is very little and the ruler moves from A towards D, or when y is vaſtly great and the ruler moves from D towards A.

§ 106. The ſame Author gives another method for diſcovering the firſt term of a ſeries that ſhall converge the ſooner the leſs y is. "Suppoſe the term where y is ſeparately of feweſt dimenſions to be Dy^l; compare it ſucceſſively with the other terms, as with $Ey^m x^s$, and obſerve where $\frac{l-m}{s}$ is found *greateſt*; and putting $\frac{l-m}{s}=n$, Ay^n will be the firſt term of a ſeries that ſhall converge the ſooner the leſs y is:" for in that caſe Dy^l and $Ey^m x^s$ will be infinitely greater than any other terms of the propoſed equation. Suppoſe $Fy^e x^k$ is any other term of the equation, and, by the ſuppoſition, $\frac{l-m}{s}$ $(=n)$ is greater than $\frac{l-e}{k}$, and conſequently, multiplying by k, you find nk greater than $l-e$, and $nk+e$ greater than l; now if for x you ſubſtitute Ay^n, then $Fy^e x^k = FA^k y^{nk+e}$, which therefore will vaniſh compared with Dy^l (ſince $nk+e$ is greater than l) when y is infinitely little. Thus therefore all the terms will vaniſh compared with Dy^l and $Ey^m x^s$ which are ſuppoſed equal; and conſequently they will give the firſt term of a ſeries that will converge the ſooner the leſs y is.

§ 107. If you obſerve "when $\frac{l-m}{s}$ is found *leaſt* of all, and ſuppoſe it equal to n, then will Ay^n be the firſt term of a ſeries that will converge the ſooner the *greater* y is." For in that caſe Dy^l and Ey^mx^s will be infinitely greater than Fy^ex^k, becauſe $\frac{l-m}{s}$ $(=n)$ being leſs than $\frac{l-e}{k}$, it follows that nk is leſs than $l-e$, and $nk+e$ leſs than l, and conſequently Fy^ex^k $(=FA^ky^{nk+e})$ vaſtly leſs than Dy^l, when y is very great.

After the ſame manner, if you compare any term Dy^lx^b, where both x and y are found, with all the other terms, and obſerve where $\frac{l-m}{s-b}$ is found *greateſt* or *leaſt*, and ſuppoſe $\frac{l-m}{s-b}=n$, then may Ay^n be the firſt term of a converging ſeries. For ſuppoſing that Fy^ex^k is any other term of the equation, if $\frac{l-m}{s-b}$ $(=n)$ is greater than $\frac{l-e}{k-b}$, then ſhall $nk-nb$ be greater than $l-e$, and $nk+e$ greater than $l+nb$. But $nk+e$ are the dimenſions of y in Fy^ex^k when $x=Ay^n$, and $l+nb$ are the dimenſions of y in Ey^mx^s, therefore Fy^ex^k is of more dimenſions of y than Ey^mx^s, and therefore vaniſhes compared to it when y is ſuppoſed infinitely little. In the ſame manner, if $\frac{l-m}{s-b}$ is leſs than $\frac{l-e}{k-b}$, then will Ey^mx^s be infinitely greater than Fy^ex^k, when y is infinite.

§ 108. When the firſt term (Ay^n) of the ſeries is found by the preceding method, then by ſuppoſing $x = Ay^n + p$, and ſubſtituting this binomial and its powers for x and its powers, there will ariſe an equation for determining p the ſecond term of the ſeries. This new equation may be treated in the ſame manner as the equation of x, and by the Rule of § 103, the terms that are to be compared in order to obtain a near value of p, may be diſcovered; by means of which terms p may be found: which ſuppoſe equal to By^{n+r}, then by ſuppoſing $p = By^{n+r} + q$, the equation may be transformed into one for determining q the third term of the ſeries, and by proceeding in the ſame manner you may determine as many terms of the ſeries as you pleaſe; finding $x = Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r}$, &c. where the dimenſions of y aſcend or deſcend according as r is poſitive or negative; and always "in arithmetical progreſſion, that this value of x being ſubſtituted for it in the propoſed equation, the terms involving y and its powers may fall in with one another, ſo that more than one may always involve the ſame dimenſion of y, which may mutually deſtroy each other and make the whole equation vaniſh, as it ought to do."

It is obvious that as the dimenſions of y in $Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r}$, &c. are in an arithmetical progreſſion whoſe difference is r, the ſquare, cube, or any power s of $Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r} +$ *&c.* will conſiſt of terms wherein the dimenſions of y will conſtitute an arithmetical progreſſion having the ſame common difference r; for theſe dimenſions will be sn, $sn + r$, $sn + 2r$, $sn + 3r$, &c. Therefore, if in any term $Ey^m x^s$ you ſubſtitute for x the ſeries $Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r}$, &c. the terms of the ſeries expreſſing $Ey^m x^s$ will conſiſt of theſe dimenſions of y, viz. $m + sn$, $m + sn + r$, $m + sn + 2r$, $m + sn + 3r$, &c. and by a like ſubſtitution in any other term as $Fy^e x^k$, the dimenſions of y will be $e + nk$, $e + nk + r$, $e + nk + 2r$, $e + nk + 3r$, &c. The former ſeries of indices muſt coincide with the latter ſeries, that the terms in which they are found may be compared together, and be found equal with oppoſite ſigns ſo as to deſtroy one another, and make the whole equation vaniſh.

The firſt ſeries conſiſts of terms ariſing by adding ſome multiple of r to $m + sn$, the latter by adding ſome multiple of r to $e + nk$; and that theſe may coincide, ſome multiple of r added to $m + sn$ muſt be equal to ſome other multiple of r added to $e + nk$. From which it appears that the difference of $m + sn$ and $e + nk$ is always a multiple of r; and conſequently that r is a diviſor of the difference of dimenſions of y in the terms $Ey^{m}x^{s}$ and $Fy^{e}x^{k}$, ſuppoſing $x = Ay^{n}$. It follows therefore "that r is a common diviſor of the differences of the dimenſions of y in the terms of the equation, when you have ſubſtituted Ay^{n} for x in all the terms." And if r be aſſumed equal to the *greateſt* common diviſor (excepting ſome caſes afterward to be mentioned you will have the true form of a ſeries for x. And now the dimenſions y^{n}, y^{n+r}, y^{n+2r}, y^{n+3r}, &c. being known, there remains only, by calculation, to determine the general coefficients A, B, C, D, &c. in order to find the ſeries $Ay^{n} + By^{n+r} + Cy^{n+2r} + Dy^{n+3r} +$ &c. $= x$.

§ 109. This leads us to Sir *Iſaac Newton's ſecond general method* of ſeries; which conſiſts in aſſuming a ſeries with undetermined coefficients expreſſing x, as $Ay^{n} + By^{n+r} + Cy^{n+2r} +$ &c. where A, B, C, &c. are ſuppoſed as yet unknown, but n and r are diſcovered by what we have already demonſtrated; and ſubſtituting this every where for x, you muſt ſuppoſe, in the new equation that ariſes, the ſum of all the terms that involve the ſame dimenſion of y to vaniſh, by which means you will obtain particular equations, the *firſt* of which will give A, the *ſecond* B, the *third* C, &c. and theſe values being ſubſtituted in the aſſumed ſeries for A, B, C, &c. the ſeries of x will be obtained as far as you pleaſe.

Let us apply, for example, this method to the equation (of § 98) $x^3 + a^2x - 2a^3 + ayx - y^3 = 0$. Suppoſe it is required to find a ſeries converging the ſooner the leſs y is: its firſt term (by § 99 or 102) is found to be a, ſo that $n = 0$. Subſtitute a for x in the equation, and the terms become $a^3 + a^3 - 2a^3 + a^2y - y^3$, and the differences of the indices are 0, 1, 2, 3; whoſe greateſt common meaſure is 1, ſo that $r = 1$. Aſſume therefore $x = A + By + Cy^2 + Dy^3$, &c. and ſubſtitute theſe ſeries for x in the equation. Then

$$\begin{array}{rcllll}
x^3 & = & A^3 + 3A^2By & +\,3AB^2y^2 & +\;\;B^3y^3 & +\,\&c. \\
 & & & +\,3A^2Cy^2 & +\,3A^2Dy^3 & +\,\&c. \\
 & & & & +\,6ABCy^3 & +\,\&c. \\
+\,a^2x & = & a^2A + a^2By & +\;a^2Cy^2 & +\;\;a^2Dy^3 & +\,\&c. \\
+\,ayx & = & \quad\;\; aAy & +\;aBy^2 & +\;\;aDy^3 & +\,\&c. \\
-\,2a^3 & = & -\,2a^3 & & & \\
-\,y^3 & = & \ldots\ldots\ldots\ldots & & -\,1 \times y^3. &
\end{array}$$

Now ſince $x^3 + a^2x + ayx - 2a^3 - y^3 = 0$, it follows that the ſum of theſe ſeries involving y muſt vaniſh. But that cannot be if the coefficient of every particular term does not vaniſh. For every term where y is infinitely little, is infinitely greater than the following terms, ſo that

if

if every term does not vaniſh of itſelf, the addition or ſubtraction of the following terms which are infinitely leſs than it, or of the preceding terms which are infinitely greater, cannot deſtroy it; and therefore the whole cannot vaniſh. It appears therefore that $A^3 + a^2A - 2a^3 = 0$, is an equation for determining A, and gives $A = a$.

In order to determine B, you muſt ſuppoſe the ſum of the coefficients affecting y to vaniſh, *viz.* $\overline{3A^2B + a^2B + aA} \times y = 0$, or, ſince $A = a$, $4a^2By + a^2y = 0$, and $B = -\frac{1}{4}$.

To determine C, in the ſame manner ſuppoſe $3AB^2y^2 + 3A^2Cy^2 + a^2Cy^2 + aBy^2 = 0$, or, ſubſtituting for A and B their values already found, $\frac{3ay^2}{16} + 4a^2Cy^2 - \frac{ay^2}{4} = 0$, and conſequently $C = \frac{1}{64a}$. And, by proceeding in the ſame manner, $D = \frac{131}{512a^2}$, ſo that $x = a - \frac{1}{4}y + \frac{1}{64a}y^2 + \frac{131}{512a^2}y^3$, &c. as we found before in § 99.

§ 110. By this method you may transfer ſeries from one undetermined quantity to another, and obtain Theorems for the *reverſion* of ſeries.

Suppoſe that $x = ay + by^2 + cy^3 + dy^4 + \&c.$ and it is required to expreſs y by a ſeries conſiſting of the powers of x. It is obvious that when x is very little, y is alſo very little, and that in order to determine the firſt term of the ſeries, you need only aſſume $x = ay$. And therefore $y = \frac{x}{a}$; ſo that $n = 1$. By ſubſtituting $\frac{x}{a}$ for y, you find the dimenſions of x in the terms will be 1, 2, 3, 4, &c. ſo that $r = 1$ alſo. You may therefore aſſume $y = Ax + Bx^2 + Cx^3 + Dx^4 +$ &c. And by the ſubſtitution of this value of y you will find

$$\begin{array}{llll} ay = & aAx + aBx^2 & + aCx^3 & + \&c. \\ by^2 = & bA^2x^2 & + 2bABx^3 & + \&c. \\ cy^3 = & & cA^3x^3 & + \&c. \\ \&c. & & \&c. & \end{array}$$

But the firſt term being already found to be $\frac{x}{2}$, you have $A = \frac{1}{a}$; and ſince $aB + bA^2 = 0$, it follows that $B = -\frac{b}{a^3}$. After the ſame manner you will find $C = \frac{2b^2 - ac}{a^4}$. Whence

$$y = \frac{x}{a} - \frac{b}{a^3}x^2 + \frac{2b^2 - ac}{a^4}x^3 + \&c.$$

§ 111. Suppoſe again you have $ax + bx^2 + cx^3 + dx^4 +$ &c. $= gy + by^2 + iy^3 + ky^4$, &c. to find x in terms of y. You will eaſily ſee, by § 103, that the firſt term of the ſeries for x is $\frac{yy}{a}$, that $n = 1$, $r = 1$. Therefore aſſume $x = Ay + By^2 + Cy^3$, &c. and by ſubſtituting this value for x and bringing all the terms to one ſide, you will have

$$\begin{array}{llll}
ax = aAy & + aBy^2 & + aCy^3 & + \&c. \\
bx^2 = & bA^2y^2 & + 2bABy^3 & + \&c. \\
cx^3 = & & cA^3y^3 & + \&c. \\
\&c. & & \&c. & \\
 & & & \\
-gy = -gy & & & \\
-by^2 = \ldots\ldots & -by^2 & & \\
-iy^3 = \ldots\ldots & \ldots\ldots & -iy^3 & \\
\&c. & & \&c. &
\end{array}$$

From whence we ſee, firſt, that $aA = g$, and $A = \frac{g}{a}$. 2°. That $aB + bA^2 - b = 0$, and $B = \frac{b}{a} - \frac{bg^2}{a^3}$. 3°. That $aC + 2bAB + cA^3 - i = 0$, and therefore $C = \frac{i - 2bAB - cA^3}{a}$. And thus the the firſt terms of the ſeries $Ay + By^2 + Cy^3$, &c. are known*.

§ 112. Before we conclude it remains to clear a difficulty in this method that has embarraſſed ſome late ingenious writers, concerning "the value of r to be aſſumed when *two* or *more* of the values of the firſt term of a ſeries for expreſſing x are found equal;" a correction of the preceding Rule being neceſſary in that caſe. And the author of that correction having only collected it from experience, and given it us

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* See Mr. *De Moivre*, in *Phil. Tranſ.* 240.

with

without proof, it is the more neceſſary to demonſtrate it here.

It is to be obſerved then, that in order that the ſeries $Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r} +$ &c. may expreſs x, it is not only neceſſary that when it is ſubſtituted for x in the propoſed equation $Dy^l + Ey^m x^s + Fy^e x^k = 0$, the indices $m + ns$, $m+ns+r$, $m + ns + 2r$, &c. ſhould fall in with the indices $e + nk$, $e + nk + r$, $e + nk + 2r$, &c. in order that the terms may be compared together to determine the coefficients A, B, C, &c. but it is alſo neceſſary, that in the particular equations for determining any of thoſe coefficients, as B for example, thoſe terms that involve B ſhould not deſtroy each other. Thus the equation $3A^2B - 3A^2B - aA = 0$ can never determine B, becauſe $3A^2B - 3A^2B = 0$, and thus B exterminates itſelf out of the equation; beſides the contradiction ariſing from $-A = 0$, when A perhaps has been determined already to be equal to ſome real quantity.

In order to know how to evite this abſurdity, let us ſuppoſe that the firſt order of terms in the propoſed equation are, as before, Dy^l, $Ey^m x^s$, &c. and if Ay^n is found to be the firſt term of a ſeries for x, then the dimenſions of y in the firſt order of terms, ariſing by ſubſtituting in them Ay^n for x, will be $m + ns$, and the dimenſions of y ariſing by ſubſtituting

$Ay^n + By^{n+r} + Cy^{n+2r}$, &c. for x will be $m+ns$, $m+ns+r$, $m+ns+2r$, &c. Suppoſe that $Fy^e x^k$ is the next order of terms, and, by the ſame ſubſtitution, the dimenſions of y ariſing from it will be (becauſe $Fy^e x^k = Fy^e \times \overline{Ay^n + By^{n+r} + Cy^{n+2r} + \&c.}|^k = FA^k y^{e+nk} + kFBA^{k-1} y^{e+nk+r}$, &c.) $e+nk$, $e+nk+r$, $e+nk+2r$, &c. Now it is plain that $e+nk$ muſt coincide with ſome of the dimenſions $m+ns$, $m+ns+r$, $m+ns+2r$, &c. that the terms involving them may be compared together. And therefore, as we obſerved in § 108, r muſt be the difference of $e+nk$ and $m+ns$, or ſome diviſor of that difference. *In general*, r muſt be aſſumed ſuch a diviſor of that difference as may allow not only $e+nk$ to coincide with ſome one of the ſeries $m+ns$, $m+ns+r$, $m+ns+2r$, &c. but as may make all the indices of the other orders beſides $e+nk$ likewiſe to coincide with one of that ſeries: that is, if $Gy^f x^h$ is another term in the equation, r muſt be ſo aſſumed that the ſeries $f+nh$, $f+nh+r$, $f+nh+2r$, &c. ariſing by ſubſtituting in it $Ay^n + By^{n+r} + Cy^{n+2r}$, &c. for x, may coincide ſomewhere with the firſt ſeries $m+ns$, $m+ns+r$, $m+ns+2r$, &c. And therefore we ſaid, in § 108, "that r muſt be aſſumed ſo as to be equal to ſome common diviſor of the differences of the indices $m+ns$, $e+nk$, $f+nh$, which ariſe in the propoſed equation by ſubſtituting in it for x the firſt term already known Ay^n." For by aſſuming r equal to a common diviſor of the differences, the three ſeries

$$\begin{array}{llll} m+ns, & m+ns+r, & m+ns+2r, & m+ns+3r, \&c. \\ e+nk, & e+nk+r, & e+nk+2r, & e+nk+3r, \&c. \\ f+nb, & f+nb+r, & f+nb+2r, & f+nb+3r, \&c. \end{array}$$

will coincide with one another, ſince ſome multiples of r added to $m+ns$ will give $e+nk$ and all that follow it in the *ſecond* ſeries, and ſome multiples of r added to $m+ns$ will alſo give $f+nb$ and all that follow it in the *third* ſeries. It is alſo obvious, that, if no particular reaſon hinder it, r ought to be aſſumed equal to the *greateſt* common meaſure of theſe differences. For example, if the indices $m+ns$, $e+nk$, $f+nb$, happen to be arithmetical progreſſion, then r ought to be aſſumed equal to the common difference of the terms, and the firſt of the ſecond ſeries will coincide with the ſecond of the firſt, and the firſt of the third ſeries will coincide with the ſecond of the ſecond ſeries, and with the third of the firſt, and ſo on.

§ 113. Theſe things being well underſtood, we are next to obſerve that after you have ſubſtituted $Ay^n + By^{n+r} + Cy^{n+2r}$, &c. for x in the firſt order of terms in the equation, the terms that involve $m+ns$ dimenſions of y will deſtroy one another; for $x - Ay^n$ muſt be a diviſor of the aggregate of theſe terms, ſince they give Ay^n as one value of x: let $\overline{x - Ay^n} \times P$ repreſent that aggregate, and, ſubſtituting for x its value $Ay^n + By^{n+r} + Cy^{n+2r}$, &c. that aggregate becomes $\overline{Ay^n + By^{n+r} + Cy^{n+2r}, \&c. - Ay^n} \times P = \overline{By^{n+r} + Cy^{n+2r}, \&c.} \times P$. Now the loweſt dimenſion in $\overline{x - Ay^n} \times P$ was ſuppoſed to be $m + ns$, whence the dimenſion of P, in the ſame terms, will be $m + ns - n$, and the loweſt dimenſion in $\overline{By^{n+r} + Cy^{n+2r} + \&c.} \times P$ will be $n + r + m + ns - n = m + ns + r$. Suppoſe again that *two* values of x, determined from the firſt order of terms, are equal, and then $\overline{x - Ay^n}\big|^2$ will be a diviſor of that aggregate of the firſt order of terms. Suppoſe that aggregate now $\overline{x - Ay^n}\big|^2 \times P$, which by ſubſtitution of $Ay^n + By^{n+r} + Cy^{n+2r}$, &c. for x will become $\overline{By^{n+r} + Cy^{n+2r} + \&c.}\big|^2 \times P$, in which the loweſt term will now be of $m + ns$ dimenſions, ſince in $\overline{x - Ay^n}\big|^2 \times P$ the loweſt term is ſuppoſed of $m + ns$ dimenſions; and conſequently, in theſe terms, the dimenſion of P itſelf is $m + ns - 2n$.

In general, if the number of values of x ſuppoſed equal to Ay^n be p, then muſt $\overline{x - Ay^n}\big|^p$ be a diviſor of the aggregate of the terms of the firſt order. And that aggregate being expreſſed by $\overline{x - Ay^n}\big|^p \times P$, in the loweſt terms, the dimenſions of y in P will be $m+ns-pn$, that in $\overline{x-Ay^n}\rvert^{p}$ they may be $m+ns$, as we always ſuppoſe. Subſtitute in $\overline{x-Ay^n}\rvert^{p} \times P$ for $x-Ay^n$ its value $By^{n+r}+Cy^{n+2r}+$ &c. and in the reſult $\overline{By^{n+r}+Cy^{n+2r}+\&c.}\rvert^{p} \times P$ the loweſt dimenſions of y will be $pn+pr+m+ns-pn = m+ns+pr$.

§ 114. From what has been ſaid we conclude that when you have ſubſtituted for x in the firſt order of terms of the equation propoſed the ſeries $Ay^n+By^{n+r}+Cy^{n+2r}+$ &c. the firſt term of which Ay^n is known, and the values of x whoſe number is p are found equal, then the terms ariſing that involve $m+ns$, $m+ns+r$, $m+ns+2r$, &c. till you come to $m+ns+pr$, will deſtroy each other and vaniſh; ſo that the firſt term with which the terms of the ſecond order $e+nk$ can be compared muſt be that which involves $m+ns+pr$; and therefore ſuppoſing $e+nk = m+ns+pr$, or $r = \frac{e+nk-m-ns}{p}$ "the higheſt value you can give r muſt be the difference of $e+nk$ and $m+ns$ divided by p the number of equal values of the firſt term of the ſeries." If this value of r is a common meaſure of all the differences of the indices, then is it a juſt value of r; but if it is not, ſuch a value of r muſt be aſſumed, as may meaſure this and all the differences: that is, "ſuch a value as may be the greateſt common meaſure of the leaſt difference divided by p (*viz.* $\frac{e+nk-m-ns}{p}$) and of the common meaſure of all the differences." For thus the indices $m+ns$, $m+ns+r$, $m+ns+2r$, &c. will coincide with $e+nk$, $e+nk+r$, $e+nk+2r$, &c. and with $f+nh$, $f+nh+r$, $f+nh+2r$, &c. and you ſhall always have terms to be compared together ſufficient to determine B, C, D, &c. the general coefficients of the ſeries aſſumed for x.

§ 115. To all this it may be added, that if $x-Ay^n$ be a diviſor of the aggregate of the terms of the *ſecond* order $Fy^e x^k$, &c. then, by ſubſtituting for x the ſeries $Ay^n+By^{n+r}+Cy^{n+2r}$ + *&c.* there vaniſh not only as many terms of the ſeries involving $m+ns$, $m+ns+r$, $m+ns+2r$, &c. as there are equal values of the firſt term Ay^n; but the terms involving $e+nk$ dimenſions of y vaniſh alſo; and therefore it is then only neceſſary that $e+nk+r$ coincide with $m+ns+pr$, ſo that, in that caſe, you need only take $r=\frac{e+nk-m-ns}{p-1}$. And if $\overline{x-Ay^n}|^{p-1}$ be a diviſor of the aggregate of the ſecond order of terms, then the terms after ſubſtituting for x the ſeries $Ay^n+By^{n+r}+Cy^{n+2r}$ &c.) which involve $e+nk$, $e+nk+r$, $e+nk+2r$, &c. will vaniſh to the term $e+nk+\overline{p-1}\times r$; ſo that, ſuppoſing $e+nk+\overline{p-1}\times r=m+ns+pr$, you have $r=e+nk-m-ns$, that is, to the *leaſt* difference of the indices $m+ns$, $e+nk$, $f+nb$, &c. provided that difference be a meaſure of the other differences; although there may be as many values of the firſt term of the ſeries equal as there are units in p. Or, if that does not happen, r muſt be taken, as formerly, equal to the greateſt common meaſure of the differences.

§ 116. Suppoſe that the orders of terms of the equation can be expreſſed the *firſt* by $\overline{x-Ay^n}|^p\times P$, the *ſecond* by $\overline{x-Ay^n}|^q\times Q$, the *third* by $\overline{x-Ay^n}|^l\times L$, &c. and ſuppoſe that Ey^mx^s is one of the firſt, Fy^ex^k one of the ſecond, Gy^fx^h one of the third, and ſo on: then it is plain that, ſubſtituting for x the ſeries $Ay^n+By^{n+r}+Cy^{n+2r}$, &c. the loweſt term that will remain in the firſt will be $m+ns+pr$ dimenſions of y, the loweſt term that will remain in the ſecond will be of $e+nk+qr$, and the loweſt term remaining in the third of $f+nb+lr$ dimenſions of y. For by the ſame reaſoning as we uſed, in § 113, to demonſtrate that, in the firſt order of terms $\overline{x-Ay^n}|^p\times P$, the loweſt dimenſions of y are $m+ns+pr$, we ſhall find that, in the ſubſequent orders, the loweſt dimenſions of y in the terms $\overline{x-Ay^n}|^q\times Q=\overline{By^{n+r}+Cy^{n+2r}\ \&c.}|^q\times Q$

muſt be $e + nk - qn + qn + qr = e + nk + qr$, and ſo of the other terms $\overline{x - Ay^n}|^l \times L$ the loweſt dimenſions muſt be $f + nb + lr$. The indices therefore of the terms that do not vaniſh being

$$\begin{array}{l} * \ * \ m+ns+pr, \\ * \ * \ * \ * \ e+nk+qr, \\ * \ * \ * \ * \ * \ * \ f+nb+lr, \end{array}$$

if r be taken equal to $\frac{e+nk-m-ns}{p-q}$, then will $m + ns + pr$ and $e + nk + qr$ coincide: and if at the ſame time r be a diviſor of $f + nb - m - ns$, and be found in it a number of times greater than $p - l$, or if r be leſs than $\frac{f+nb-m-ns}{p-l}$, then r will be rightly aſſumed. *In general*, "take all the quotients $\frac{e+nk-m-ns}{p-q}$, $\frac{f+nb-m-ns}{p-l}$, and either the *leaſt* of theſe, or a number whoſe denominator, exceeding $p - q$ by an integer, meaſures it and all the differences $f + nb - m - ns$, gives r;" ſuppoſing p, q, and l integers. But if p, q, and l are fractions, you are to "take r ſo that it be equal to $\frac{e+nk-m-ns}{p-q+K} = \frac{f+nb-m-ns}{p-l+M}$, and ſo that K and M may be integers." Suppoſe, for example, $m + ns = \frac{7}{3}$, $p =$

$p=\frac{5}{2}$; $e+nk=\frac{10}{3}$, $q=\frac{3}{2}$; $f+nb=\frac{9}{2}$, and $l=\frac{1}{2}$: then putting ------ $(r=)$ $\frac{e+nk-m-ns}{p-q+K}=\frac{1}{1+K}=\frac{f+nb-m-ns}{p-l+M}=\frac{\frac{13}{6}}{2+M}$. $M=\frac{1}{6}+\frac{13}{6}K$; whence it is eaſily ſeen that 5 and 11 are the leaſt integers that can be aſſumed for K and M. And that $r=\frac{1}{1+K}=\frac{1}{6}$; and therefore $m+ns+pr=\frac{33}{12}$, $e+nk+qr=\frac{43}{12}$, and $f+nb+lr=\frac{55}{12}$. That is, the terms of the firſt ſeries whoſe dimenſions are $m+ns+\overline{p+K}\times r$, $m+ns+\overline{p+M}\times r$ fall in with the firſt terms of the ſecond and third ſeries reſpectively *.

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* See on this ſubject, *Colſon.* Epiſt. in Animadv. D. *Moivrei.* *Taylor* Meth. Incr. *Stirling* Lin. iij. Ord. *s'Graveſande* Append. Elem. Algebrae. *Stewart* on the Quadrature of Curves.

CHAP. XI.

Of the Rules for finding the number of impoſſible Roots in an equation.

§ 117. THE number of *impoſſible* roots in an equation may, for moſt part, be found by this

RULE.

" *Write down a ſeries of fractions whoſe denominators are the numbers in this progreſſion* 1, 2, 3, 4, 5, &c. *continued to the number which expreſſes the dimenſion of the equation. Divide every fraction in the ſeries by that which precedes it, and place the quotients in order over the middle terms of the equation. And if the ſquare of any term multiplied into the fraction that ſtands over it gives a product greater than the rectangle of the two adjacent terms, write under the term the ſign* +, *but if that product is not greater than the rectangle, write* —; *and the ſigns under the extreme terms being* +, *there will be as many* imaginary *roots as there are changes of the ſigns from* + *to* —, *and from* — *to* +.

Thus, the given equation being $x^3+px^2+3p^2x-q=0$, I divide the ſecond fraction of the ſeries $\frac{3}{1}$, $\frac{2}{2}$, $\frac{1}{3}$, by the firſt, and the third by the ſecond, and place the quotients $\frac{1}{3}$ and $\frac{1}{3}$ over the middle terms in this manner;

$$\underset{+}{x^3} \; \underset{-}{+\overset{\frac{1}{3}}{px^2}} \; \underset{+}{+\overset{\frac{1}{3}}{3p^2x}} \; \underset{+}{-q}=0.$$

Then becauſe the ſquare of the ſecond term multiplied into the fraction that ſtands over it, that is, $\frac{1}{3}\times p^2x^4$ is leſs than $3p^2x^4$ the rectangle under the firſt and third terms, I place under the ſecond term the ſign $-$: but as $\frac{1}{3}\times 9p^4x^2$ $(=3p^4x^2)$ the ſquare of the third term multiplied into its fraction is greater than *nothing*, and conſequently much greater than $-pqx^2$ the negative product of the adjoining terms, I write under the third term the ſign $+$. I write $+$ likewiſe under x^3 and $-q$ the firſt and laſt terms, and finding in the ſigns thus marked two changes, one from $+$ to $-$, and another from $-$ to $+$, I conclude the equation has two impoſſible roots.

In like manner the equation $x^3-4x^2+4x-6=0$ has two impoſſible roots;

$$\underset{+}{x^3} \; \underset{+}{-\overset{\frac{1}{3}}{4x^2}} \; \underset{-}{+\overset{\frac{1}{3}}{4x}} \; \underset{+}{-6}=0;$$

and the equation $x^{4*} - 6x^2 - 3x - 2 = 0$ the ſame number

$$\begin{array}{ccccc} & \frac{3}{8} & \frac{4}{9} & \frac{3}{8} & \\ x^{4} & * & -6x^2 & -3x & -2 = 0. \\ + & + & + & - & + \end{array}$$

For the ſeries of fractions $\frac{4}{1}$, $\frac{3}{2}$, $\frac{2}{3}$, $\frac{1}{4}$ yields, by dividing them as the Rule directs, the fractions $\frac{3}{8}$, $\frac{4}{9}$, $\frac{3}{8}$ to be placed over the terms. Then the ſquare of the ſecond term, which is *nothing*, multiplied by the fraction over it being ſtill *nothing*, and yet greater than $-6x^6$ the negative product of the adjacent terms, I write under (*) the term that is wanting, the ſign +, and proceeding as in the former examples, I conclude, from the two changes that happen in the ſeries $+ + + - +$, that the equation has two of its roots impoſſible.

The ſame way we diſcover two impoſſible roots in the equation

$$\begin{array}{cccccc} & \frac{2}{5} & \frac{1}{2} & \frac{1}{2} & \frac{2}{5} & \\ x^5 & -4x^4 & +4x^3 & -2x^2 & -5x & -4 = 0. \\ + & + & - & + & + & + \end{array}$$

When two or more terms are wanting in the equation, under the firſt of ſuch terms place the ſign $-$, under the ſecond $+$, under the third $-$, and ſo on alternately; only when the two terms to the right and left of the deficient terms have contrary ſigns, you are always to write the ſign + under the laſt deficient term.

As in the equations

$$\begin{array}{cccccc} x^5 & +ax^4 & * & * & * & +a^5=0 \\ + & + & - & + & - & + \end{array}$$

$$\text{and}\quad \begin{array}{cccccc} x^5 & +ax^4 & * & * & * & -a^5=0 \\ + & + & - & + & + & + \end{array}$$

the firſt of which has *four* impoſſible roots, and the other *two*. Thus likewiſe the equation

$$\begin{array}{cccccccc} & \frac{3}{7} & \frac{5}{9} & \frac{3}{5} & \frac{3}{5} & \frac{5}{9} & \frac{3}{7} & \\ x^7 & -2x^6 & +3x^5 & -2x^4 & +x^3 & * & * & -3=0 \\ + & - & + & - & + & - & + & + \end{array}$$

has *ſix* impoſſible roots.

Hence too we may diſcover if the imaginary roots lie hid among the affirmative, or among the negative roots. For the ſigns of the terms which ſtand over the ſigns below that change from + to — and — to +, ſhew, by the number of their variations, how many of the impoſſible roots are to be reckoned affirmative; and that there are as many negative imaginary roots as there are repetitions of the ſame ſign. As in the equation

$$\begin{array}{cccccc} x^5 & -4x^4 & +4x^3 & -2x^2 & -5x & -4=0 \\ + & + & - & + & + & + \end{array}$$

the ſigns (— + —) of the terms $-4x^4+4x^3-2x^2$ which ſtand over the ſigns + — + point-ing out two affirmative roots *, we infer that two impoſſible roots lie among the affirmative: and the three changes of the ſigns in the equation (+ — + — — —) giving three affirmative roots and two negative, the five roots will be one real affirmative, two negative, and two imaginary affirmatives. If the equation had been

$$\begin{array}{cccccc} x^5 - & 4x^4 - & 4x^3 - & 2x^2 - & 5x - & 4 = 0, \\ + & + & - & - & + & + \end{array}$$

the terms $-4x^4-4x^3$ that ſtand over the firſt variation + —, ſhew, by the repetition of the ſign —, that one imaginary root is to be reckoned negative, and the terms $-2x^2-5x$ that ſtand over the laſt variation — +, give, for the ſame reaſon, another negative impoſſible root; ſo that the ſigns of the equation (+ — — — — —) giving one affirmative root, we conclude that of the four negative roots, two are imaginary.

This always holds good unleſs, which ſometimes may happen, there are more impoſſible roots in the equation than are diſcoverable by the Rule."

This Rule hath been inveſtigated by ſeveral eminent Mathematicians in various ways; and others, ſimilar to it, invented and publiſhed †. *But the*

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* See § 19.

† See *Stirling*'s Lineae iij. Ord. *Newton*. p. 59. *Phil. Tranſ.* N° 394, 404, 408.

original Rule being, on account of its ſimplicity and eaſy application, if not preferable to all others, at leaſt the fitteſt for this place, it is ſufficient to direct the Reader where he may find the ſubject more fully treated; and to add the demonſtration our Author has given of it towards the end of his Letter to Mr. Folkes, Phil. Tranſ. N° 408, *as it depends only on what has been demonſtrated in* Chap. 5. *concerning the limits of the roots of equations.*

§ 118. Let $ax^2 \pm px \pm q = 0$ be any adfected quadratic equation; and, by § 18, *Part* I. its roots will be $\frac{1}{2a} \times \overline{\mp p \pm \sqrt{p^2 \mp 4aq}}$: whence it is plain that, the ſign of q in the given equation being $+$, the roots will be impoſſible as oft as $4aq$ is greater than p^2, or $\frac{1}{4}p^2$ leſs than $a \times q$.

§ 119. It was ſhewn, *in general* (§ 45 — 50) *that the roots of the equation* $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3}$ &c. $= 0$, *are the* limits *of the roots of the equation* $nx^{n-1} - \overline{n-1} \times Ax^{n-2} + \overline{n-2} \times Bx^{n-3}$ &c. $= 0$, *or of any equation that is deduced from it by multiplying its terms by any arithmetical progreſſion* $l \mp d$, $l \mp 2d$, $l \mp 3d$, &c. *and converſely the roots of this new equation will be the* limits *of the roots of the propoſed equation* $x^n - Ax^{n-1} + Bx^{n-2}$ &c. $= 0$.

And that *if any roots of the equation of the limits are impoſſible, there muſt be ſome roots of the propoſed equation impoſſible.*

§ 120. Let $x^3 - Ax^2 + Bx - C = 0$ be a cubic equation, and the *equation of limits* $3x^2 - 2Ax + B = 0$. If the two roots of this laſt are imaginary, there are two imaginary roots of the given equation $x^3 - Ax^2 + Bx - C = 0$, by the laſt *Art.* But, by the preceding *Art.* this happens as oft as $\frac{1}{3}A^2$ is leſs than B; and, in that caſe, the given equation has two imaginary roots.

Again, multiplying the terms of the equation by the terms of the progreſſion, $0, -1, -2, -3$, we get another *equation of the limits* $Ax^2 - 2Bx + 3C = 0$; whoſe two roots, and conſequently two roots of the given equation, are imaginary when $\frac{1}{3}B^2$ is leſs than $A \times C$.

Hence likewiſe the biquadratic $x^4 - Ax^3 + Bx^2 - Cx + D = 0$, will have two imaginary roots, if two roots of the equation $4x^3 - 3Ax^2 + 2Bx - C = 0$ be imaginary; or if two roots of the equation $Ax^3 - 2Bx^2 + 3Cx - 4D = 0$ be imaginary. But two roots of the equation $4x^3 - 3Ax^2 + 2Bx - C = 0$ muſt be imaginary, when two roots of the quadratic $6x^2 - 3Ax + B = 0$, or of the quadratic $3Ax^2 - 4Bx + 3C = 0$, are imaginary, becauſe the roots of theſe quadratic equations are the *limits* of the roots of that cubic, and for the ſame reaſon two roots of the cubic equation $Ax^3 - 2Bx^2 + 3Cx - 4D = 0$ muſt be imaginary, when the roots of the quadratic $3Ax^2 - 4Bx + 3C = 0$, or of the quadratic $Bx^2 - 3Cx + 6D = 0$ are impoſſible. Therefore two roots of the biquadratic $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ muſt be imaginary when the roots of any one of theſe three quadratic equations $6x^2 - 3Ax + B = 0$, $3Ax^2 - 4Bx + 3C = 0$, $Bx^2 - 3Cx + 6D = 0$ become imaginary; that is, when $\frac{3}{8}A^2$ is leſs than B, $\frac{4}{9}B^2$ leſs than AC, or $\frac{3}{8}C^2$ leſs than BD.

§ 121. By proceeding in the ſame manner, you may deduce from any equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3}$ &c. $= 0$, as many quadratic equations as there are terms excepting the firſt and laſt, whoſe roots muſt be all real quantities, if the propoſed equation has no imaginary roots. The quadratic deduced from the three firſt terms $x^n - Ax^{n-1} + Bx^{n-2}$ will manifeſtly have this form, $n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}$ &c. $\times x^2 - \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}$ &c. $\times Ax + \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times \overline{n-5}$ &c. $\times B = 0$, continuing the factors in each till you have as many as there are units in $n-2$. Then dividing the equation by all the factors $n-2$, $n-3$, $n-4$, &c. which are found in each coefficient, the equation will become $n \times \overline{n-1} \times x^2 - \overline{n-1} \times 2Ax + 2 \times 1 \times B = 0$, whoſe roots will be imaginary, by § 118, when $n \times \overline{n-1} \times 2 \times 4B$ exceeds $\overline{n-1}|^2 \times 4A^2$, or when B exceeds $\frac{n-1}{2n}A^2$: ſo that the propoſed equation muſt have ſome imaginary roots when B exceeds $\frac{n-1}{2n}A^2$. The quadratic equation deduced in the ſame manner from the three firſt terms of the equation $Ax^{n-1} - 2Bx^{n-2} + 3Cx^{n-3}$ &c. $= 0$, will have this form, $\overline{n-1} \times \overline{n-2} \times \overline{n-3}$ &c. $\times Ax^2 - \overline{n-2} \times \overline{n-3} \times \overline{n-4}$ &c. $\times 2Bx + \overline{n-3} \times \overline{n-4} \times \overline{n-5}$ &c. $\times 3C = 0$, which dividing by the factors common to all the terms, is reduced to $\overline{n-1} \times \overline{n-2} \times Ax^2 - \overline{n-2} \times 4Bx + 6C = 0$, whoſe roots muſt be imaginary when $\frac{2}{3} \times \frac{n-2}{n-1} \times B^2$ is leſs than AC; and therefore in that caſe ſome roots of the propoſed equation muſt be imaginary.

§ 122. In general, let $Dx^{n-r+1} - Ex^{n-r} + Fx^{n-r-1}$ be any three terms of the equation, $x^n - Ax^{n-1} + Bx^{n-2}$ &c. $= 0$, that immediately follow one another; multiply the terms of this equation firſt by the progreſſion n, $n-1$, $n-2$, &c. then by the progreſſion $n-1$, $n-2$, $n-3$, &c. then by $n-2$, $n-3$, $n-4$, &c. till you have multiplied by as many progreſſions as there are units in $n-r-1$: then multiply the terms of the equation that ariſes, as often by the progreſſion 0, 1, 2, 3, *&c.* as there are units in $r-1$, and you will at length arrive at a quadratic of this form;

$\overline{n-r+1} \times \overline{n-r} \times \overline{n-r-1} \times \overline{n-r-2}$ &c. $\times \overline{r-1} \times \overline{r-2} \times \overline{r-3} \times \overline{r-4}$ &c. $\times Dx^2$

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$-\overline{n-r}\times\overline{n-r-1}\times\overline{n-r-2}\times\overline{n-r-3}$ &c. $\times r\times\overline{r-1}\times\overline{r-2}\times\overline{r-3}$ &c. $\times Ex$ $+\overline{n-r-1}\times\overline{n-r-2}\times\overline{n-r-3}\times\overline{n-r-4}$ &c. $\times\overline{r+1}\times r\times\overline{r-1}\times\overline{r-2}$ &c. $\times F=0$: and dividing by the factors $n-r-1$, $n-r-2$, *&c.* and $r-1$, $r-2$, *&c.* which are found in each coefficient, this equation will be reduced to $\overline{n-r+1}\times\overline{n-r}\times 2\times 1\times Dx^2-\overline{n-r}\times 2\times r\times 2Ex+2\times 1\times\overline{r+1}\times rF=0$, whoſe roots muſt be imaginary, by § 118, when $\frac{n-r}{n-r+1}\times\frac{r}{r+1}\times E^2$ is leſs than DF. From which it is manifeſt, that if you divide each term of this ſeries of fractions $\frac{n}{1}$, $\frac{n-1}{2}$, $\frac{n-2}{3}$, $\frac{n-3}{4}$, &c. $\frac{n-r+1}{r}$, $\frac{n-r}{r+1}$, by that which precedes it, and place the quotients above the terms of the equation $x^n-Ax^{n-1}+Bx^{n-2}-Cx^{n-3}$ &c. $=0$, beginning with the ſecond: then if the ſquare of any term multiplied by the fraction over it be found leſs than the product of the adjacent terms, ſome of the roots of that equation muſt be imaginary quantities.

§ 123. An equation may have impoſſible roots although none are diſcovered by the Rule: becauſe, "though *real* roots in the *given equation* always give *real* roots in the *equation of limits*; yet it does not follow, *converſely*, that when the roots of the *equation of limits* are real, thoſe of the equation from which it is produced muſt be ſuch likewiſe. Thus the *cubic*

$$\left.\begin{matrix} x^3 - 2m \\ - q \end{matrix}\right\} \times x^2 \left.\begin{matrix} + m^2 \\ + 2qm \\ + n \end{matrix}\right\} \times x - q \times \overline{m^2 + n} = 0,$$

has two of its roots imaginary, $m + \sqrt{-n}$, $m - \sqrt{-n}$, the third being $+q$: and yet in the *equation of limits* $3x^2 - \overline{4m + 2q} \times x^2 + m^2 + 2qm + m = 0$, if $\overline{m-q}|^2$ exceeds $3n$, the roots of the equation of limits will be real. Or if the other equation of limits $\overline{2m + q} \times x^2 - 2 \times \overline{m^2 + 2qm + n} \times x + 3q \times \overline{m^2 + n} = 0$ is found by multiplying by the progreſſion $0, -1, -2, -3$; it will have its roots real as oft as $\overline{m^2 + 2qm + n}|^2$ exceeds $\overline{2m + q} \times 3q \times \overline{m^2 + n}$. And the like may be ſhewn of higher equations.

§ 124. The reaſon why this Rule, and perhaps every other that depends on the compariſon of the ſquare of a term with the rectangles of the terms on either ſide of it, muſt ſometimes fail to diſcover the impoſſible roots, may appear likewiſe from this conſideration; that the number of ſuch compariſons being always leſs by *unit* than the number of the quantities q, m, n, &c. in the general equation; they cannot include and fix the relations of theſe quantities, on which the ratio of *greater* or *leſſer* inequality of the ſquares and rectangles depends; no more than equations fewer in number than the quantities ſought can furniſh a determinate ſolution of a problem.

CHAP. XII.

Containing a general demonſtration of Sir *Iſaac Newton*'s Rule for finding the ſums of the powers of the roots of an equation *.

LET the equation be $\overline{x-a}\times\overline{x-b}\times\overline{x-c}\times\overline{x-d}\times$ &c. $=0$, or,

$$\left.\begin{array}{l} x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} \ldots \\ \ldots - Ix^3 + Kx^2 - Lx + M \end{array}\right\} = 0.$$

It is known that $A = a + b + c + d +$ &c. $B = ab + ac + ad + bc + bd + cd +$ &c. $C = abc + abd + bcd +$ &c. $D = abcd +$ &c. the *parts* or *terms* of the coefficients A, B, C, D, &c. being of 1, 2, 3, 4, *&c. dimenſions*; that is, containing as many *roots* or *factors* as there are terms of the equation preceding them, reſpectively.

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* See *Arith. Univerſ.* pag. 157. And *Chap.* II. § 15—17, of this *Part*.

CASE

CASE I.

Let r be an *index equal* to n, or *greater* than n, then, multiplying the equation by x^{r-n}, and ſubſtituting ſucceſſively a, b, c, d, &c. for x, you obtain

$$\left.\begin{array}{l} a^r - Aa^{r-1} + Ba^{r-2} - Ca^{r-3} \dots \\ \dots - La^{r-n+1} + Ma^{r-n} \end{array}\right\} = 0.$$

$$\left.\begin{array}{l} b^r - Ab^{r-1} + Bb^{r-2} - Cb^{r-3} \dots \\ \dots - Lb^{r-n+1} + Mb^{r-n} \end{array}\right\} = 0,$$

$$\left.\begin{array}{l} c^r - Ac^{r-1} + Bc^{r-2} - Cc^{r-3} \dots \\ \dots - Lc^{r-n+1} + Mc^{r-n} \end{array}\right\} = 0,$$

&c.

Whence, by tranſpoſition and addition, this *Theorem* reſults, that, in this caſe, "the ſum of the powers of the roots, of the exponent r, is equal to the ſum of their powers of the exponent $r-1$ multiplied by A, *minus* the ſum of their powers of the exponent $r-2$ multiplied by B, $+$ the ſum of thoſe of the exponent $r-3$ multiplied by C, and ſo on."

It remains to find the ſums of the powers of the roots, when the exponents are *leſs* than n the exponent of the equation.

CASE II.

If r is leſs than n, and H be the coefficient in the equation, of the *dimenſions* r; that is, if H be taken ſo that the number of terms preceding it in the equation be equal to r, or the number of *factors* in its parts *abcdefgh*, *abcdefgi*, &c, equal to r, then the Theorem may be expreſſed in the following manner.

$$a^r + b^r + c^r + d^r + \&c.$$

$$= \left\{\begin{matrix} +a^{r-1} \\ +b^{r-1} \\ +c^{r-1} \\ +d^{r-1} \\ +\&c. \end{matrix}\right\} \times A \left.\begin{matrix} -a^{r-2} \\ -b^{r-2} \\ -c^{r-2} \\ -d^{r-2} \\ -\&c. \end{matrix}\right\} \times B \left.\begin{matrix} +a^{r-3} \\ +b^{r-3} \\ +c^{r-3} \\ +d^{r-3} \\ +\&c. \end{matrix}\right\} \times C \ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots - r \times H.$$

The caſe when $r = n - 1$ is eaſily demonſtrated; for, dividing the equation by x, we have

$$x^{n-1} - Ax^{n-2} + Bx^{n-3} \ldots\ldots - L + \frac{M}{x} = 0.$$

Whence

$$a^{n-1} - Aa^{n-2} + Ba^{n-3} \ldots\ldots - L + \frac{M}{a} = 0,$$

$$b^{n-1} - Ab^{n-2} + Bb^{n-3} \ldots\ldots - L + \frac{M}{b} = 0,$$

$$c^{n-1} - Ac^{n-2} + Bc^{n-3} \ldots\ldots - L + \frac{M}{c} = 0,$$

&c. &c.

and (becauſe $L = \frac{M}{a} + \frac{M}{b} + \frac{M}{c} + \frac{M}{d} + \&c.$) we ſhall have $a^{n-1} + b^{n-1} + c^{n-1} + \&c.$

$$= \left\{\begin{matrix} +a^{n-2} \\ +b^{n-2} \\ +c^{n-2} \\ +\&c. \end{matrix}\right\} \times A \left.\begin{matrix} -a^{n-3} \\ -b^{n-3} \\ -c^{n-3} \\ -\&c. \end{matrix}\right\} \times B \left.\begin{matrix} +a^{n-4} \\ +b^{n-4} \\ +c^{n-4} \\ +\&c. \end{matrix}\right\} \times C \ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots + \overline{n-1} \times L.$$

When $r = n - 2$, the demonſtration is derived from hence, that $\overline{a^2 + b^2 + c^2 + d^2 + \&c.} = A^2 - 2B$ (*pag.* 142.) as follows.

By § 32. transform the given equation, *viz.*

$$\left.\begin{matrix} x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} \ldots \\ \ldots - Ix^3 + Kx^2 - Lx + M \end{matrix}\right\} = 0 \text{ into}$$

the equation

$$\left.\begin{matrix} z^n - \frac{L}{M}z^{n-1} + \frac{K}{M}z^{n-2} - \frac{I}{M}z^{n-3} \ldots \\ \ldots - \frac{C}{M}z^3 + \frac{B}{M}z^2 - \frac{A}{M}z + \frac{1}{M} \ldots \end{matrix}\right\} = 0;$$

the roots $\alpha, \beta, \gamma, \delta$, &c. of which new equation ſhall be reſpectively equal to the reciprocals $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$, &c. of the roots of the original equation.

Divide now the original equation by x^2, and in the quotient ſubſtitute for x the roots a, b, c, d, &c. ſucceſſively, ſo you ſhall have

$$\left.\begin{matrix} a^{n-2} - Aa^{n-3} + Ba^{n-4} - Ca^{n-5} \ldots \\ \ldots - Ia + K - \frac{L}{a} + \frac{M}{a^2} \end{matrix}\right\} = 0,$$

$$\left.\begin{matrix} b^{n-2} - Ab^{n-3} + Bb^{n-4} - Cb^{n-5} \ldots \\ \ldots - Ib + K - \frac{L}{b} + \frac{M}{b^2} \end{matrix}\right\} = 0,$$

$$\left.\begin{matrix} c^{n-2} - Ac^{n-3} + Bc^{n-4} - Cc^{n-5} \ldots \\ \ldots - Ic + K - \frac{L}{c} + \frac{M}{c^2} \end{matrix}\right\} = 0,$$

&c.

Add all theſe equations together, and for $n - 2$ ſubſtitute its value r, and it will be

ar

$$\left.\begin{array}{l} a^r - a^{r-1} \\ + b^r - b^{r-1} \\ + c^r - c^{r-1} \\ + \&c. - \&c. \end{array}\right\} \times A \left.\begin{array}{l} a^{r-2} \\ + b^{r-2} \\ + c^{r-2} \\ + \&c. \end{array}\right\} \times B \,.\,.\,,\,.$$

$$..\left\{\begin{array}{l} -a \\ -b \\ -c \\ -\&c. \end{array}\right\} \times I + nK - \frac{L}{M} \times L + M \times \left\{\begin{array}{l} +\alpha^2 \\ +\beta^2 \\ +\gamma^2 \\ +\&c. \end{array}\right\} = 0.$$

But by the principle adduced from *pag.* 142, $\alpha^2 + \beta^2 + \gamma^2 + \&c. = \frac{L^2}{M^2} - \frac{2K}{M}$: wherefore, by multiplication and tranſpoſition, it will follow that

$$2K - \frac{L}{M} \times L + M \times \left\{\begin{array}{l} +\alpha^2 \\ +\beta^2 \\ +\gamma^2 \\ +\&c. \end{array}\right\} = 0.$$

Which equation being ſubtracted from the preceding, there remains

$$\left.\begin{array}{l} a^r - a^{r-1} \\ + b^r - b^{r-1} \\ + c^r - c^{r-1} \\ + \&c. - \&c. \end{array}\right\} \times A \left.\begin{array}{l} + a^{r-2} \\ + b^{r-2} \\ + c^{r-2} \\ + \&c. \end{array}\right\} \times B, \ldots.$$

$$\ldots\left\{\begin{array}{l} -a \\ -b \\ -c \\ -\&c. \end{array}\right\} \times I + r \times K = 0.$$ Which was to be proved.

But to ſhew it *univerſally*, we may uſe the following LEMMA:

" That if A is the coefficient of one dimenſion, or the coefficient of the ſecond term, in an equation, G any other coefficient, H the coefficient next after it; the difference of the dimenſions of G and A being $r-2$: if likewiſe $A' \times G'$ repreſent the ſum of all thoſe terms of the product $A \times G$ in which the ſquare of any root, as a^2, or b^2, or c^2, &c. is found; then will $A' \times G' = AG - rH$."

This is a particular caſe of *Prop.* VI. concerning the *impoſſible roots* in *Phil. Tranſ.* N° 408; which, by continuing the Table of Equations in *pag.* 140, and obſerving how the coefficients are formed, may be thus demonſtrated.

Let the coefficient of a term of the equation, as D ($= abcd + abce + abcf$, &c. $+ bcde + bcdf$ &c.) be multiplied by A ($= a + b + c + d +$ &c.) and, in the product $A \times D$, ſetting aſide all the terms, $A' \times D'$, in which a^2, b^2, c^2, &c. are found, any one of the remaining terms will ariſe as often as there are factors in the terms of the following coefficient E. Thus the term $abcde$ will ariſe *five* times: becauſe it is made up of any one of the five roots (or terms of A) a, b, c, d, e, multiplied into the other four that make a term of D: the like is true of every other term, as $abcdf$, $bcdef$, &c. each of which will ariſe *five* times in the product $A \times D$. And the ſum of theſe terms $abcde + abcdf +$ &c. making up the coefficient E, it follows that

$A\times D - A'\times D' = 5E$, or $A'\times D' = AD - 5E$. And the ſame holds of any two coefficients G, H, whoſe dimenſions are $r-1$ and r reſpectively.

To apply this to the preſent purpoſe, it is to be obſerved that, in each of the coefficients A, B, C, D, &c. except the laſt M, which is the product of all the roots a, b, c, d, &c. we may diſtinguiſh two ſeveral portions or members, in one of which any particular root, as a, is contained, but in the whole remaining portion of the ſame coefficient, that particular root (a) is wholly abſent. Now if, for brevity's ſake, we denote that portion of any coefficient wherein any root, as a, is contained, by annexing the ſymbol of the ſaid root with the ſign + in an *uncus* to the ſymbol, as G, of that coefficient (thus $G^{(+a}$;) and if we denote the remaining portion of the ſame coefficient, from which the ſame root a is totally abſent by annexing the ſymbol of the ſaid root with the ſign — in an *uncus* to the ſymbol G of the ſame coefficient (thus $G^{(-a}$) it will appear that (if G be any coefficient and H the following coefficient)

$$G = G^{(+a} + G^{(-a} \text{ and } H^{(+a} = aG^{(-a},$$
$$G = G^{(+b} + G^{(-b} \text{ and } H^{(+b} = bG^{(-b},$$
$$\text{\&c.} \qquad\qquad\qquad \text{\&c.}$$

Divide now the equation propoſed by x^{n-r}, and it will become

$$\left.\begin{array}{l} x^r - Ax^{r-1} + Bx^{r-2} - Cx^{r-3} \ldots\ldots \\ \ldots + Gx - H + \frac{I}{x} - \frac{K}{x^2} + \frac{L}{x^3} - \frac{M}{x^{n-r}} \end{array}\right\} = 0,$$

in which ſubſtituting a, b, c, &c. ſucceſſively for x, we obtain

$$\left.\begin{array}{l} a^r - Aa^{r-1} + Ba^{r-2} - Ca^{r-3} \ldots\ldots \\ \ldots + Ga - H + \frac{I}{a} - \frac{K}{a^2} + \frac{L}{a^3} - \frac{M}{a^{n-r}} \end{array}\right\} = 0,$$

$$\left.\begin{array}{l} b^r - Ab^{r-1} + Bb^{r-2} - Cb^{r-3} \ldots\ldots \\ \ldots + Gb - H + \frac{I}{b} - \frac{K}{b^2} + \frac{L}{b^3} - \frac{M}{b^{n-r}} \end{array}\right\} = 0,$$

$$\left.\begin{array}{l} c^r - Ac^{r-1} + Bc^{r-2} - Cc^{r-3} \ldots\ldots \\ \ldots + Gc - H + \frac{I}{c} - \frac{K}{c^2} + \frac{L}{c^3} - \frac{M}{c^{n-r}} \end{array}\right\} = 0.$$

But, by the notation here uſed, and explained as above,

$$\begin{array}{lllll}
Ga = & aG^{(+a} & + aG^{(-a} & & \\
-H = & & - aG^{(-a} & - H^{(-a} & \\
+\frac{I}{a} = & & & + H^{(-a} & + \frac{I}{a}^{(-a} \\
-\frac{K}{a^2} = & - \frac{I}{a}^{(-a} & - \frac{K}{a^2}^{(-a} & & \\
+\frac{L}{a^3} = & & + \frac{K}{a^2}^{(-a} & + \frac{L}{a^3}^{(-a} & \\
-\frac{M}{a^{n-3}} = & & & - \frac{L}{a^2}^{(-a} & \text{\&c.}
\end{array}$$

Whence

$$Ga - H + \frac{I}{a} - \frac{K}{a^2} + \frac{L}{a^3} - \frac{M}{a^{n-r}} = aG^{(+a}$$

$$Gb - H + \frac{I}{b} - \frac{K}{b^2} + \frac{L}{b^3} - \frac{M}{b^{n-r}} = bG^{(+b}$$

$$Gc - H + \frac{I}{c} - \frac{K}{c^2} + \frac{L}{c^3} - \frac{M}{c^{n-r}} = cG^{(+c}$$

&c.

And the ſum of theſe $= aG^{(+a} + bG^{(+b} + cG^{(+c} +$ *&c.* $=$ (by this notation) $A' \times G' =$ (by the lemma)

$$\left.\begin{matrix} +a \\ +b \\ +c \\ +\&c. \end{matrix}\right\} \times G - rH.$$

Compare this laſt concluſion with that which followed from dividing the propoſed equation by x^{n-r}, and ſubſtituting for x the roots a, b, c, &c. and you will have

$$\left.\begin{matrix} \left.\begin{matrix} a^r & -a^{r-1} \\ +b^r & -b^{r-1} \\ +c^r & -c^{r-1} \\ +\&c., & -\&c. \end{matrix}\right\} \times A \left.\begin{matrix} +a^{r-2} \\ +b^{r-2} \\ +c^{r-2} \\ +\&c. \end{matrix}\right\} \times C \ldots\ldots \\ \ldots\ldots\ldots\ldots \left\{\begin{matrix} +a \\ +b \\ +c \\ +\&c. \end{matrix}\right\} \times G - rH \end{matrix}\right\} = 0,$$

which was to be demonſtrated.

From theſe two Theorems Sir *Iſaac Newton's* Rule manifeſtly follows.

But, to illuſtrate the reaſoning here uſed by ſome examples: ſuppoſe $r = 3$, then we are to take C for H, becauſe three terms only precede C in the equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = 0$; and we are to prove that

$$\left.\begin{array}{l} a^3 \\ +b^3 \\ +c^3 \\ +d^3 \\ +\&c. \end{array}\right\} = \left.\begin{array}{l} a^2 \\ +b^2 \\ +c^2 \\ +d^2 \\ +\&c. \end{array}\right\} \times A \left.\begin{array}{l} -a \\ -b \\ -c \\ -d \\ -\&c. \end{array}\right\} \times B + 3C.$$

That this may appear, obſerve that $a^3 + b^3 + c^3 + d^3 + \&c. = \overline{a^2 + b^2 + c^2 + d^2} + \&c. \times \overline{a + b + c + d} + \&c. - a^2 \times \overline{b + c + d} + \&c. - b^2 \times \overline{a + c + d} + \&c. - c^2 \times \overline{a + b + d} + \&c. - d^2 \times \overline{a + b + c} + \&c. - \&c. =$ (becauſe $A'B' = a \times \overline{ab + ac + ad} + \&c. + b \times \overline{ab + bc + bd} + \&c. + c \times \overline{ac + bc + dc} + \&c. + d \times \overline{ad + bd + cd} + \&c. + \&c.$) $= \overline{a^2 + b^2 + c^2 + d^2} + \&c. \times A - A'B'$ (by the Lemma) $= \overline{a^2 + b^2 + c^2 + d^2} + \&c. \times A - AB + 3C$.

In like manner, $\overline{a^4 + b^4 + c^4 + d^4} + \&c. = \overline{a^3 + b^3 + c^3 + d^3} + \&c. \times \overline{a + b + c + d} + \&c. - \overline{a^2 + b^2 + c^2 + d^2} + \&c. \times \overline{ab + ac + ad + bc + bd + cd} + \&c. + a^2 \times \overline{bc + bd + cd} + \&c. + b^2 \times \overline{ac + ad + cd} + \&c.$

$$+\ \&c. + c^2 \times \overline{ab+ad+bd} + \&c. + d^2 \times \overline{ab+ac+bc}$$
$$+\ \&c. + \&c. = \overline{a^3+b^3+c^3+d^3} + \&c. \times A$$
$$-\ \overline{a^2+b^2+c^2+d^2} + \&c. \times B + AC =$$
$$\overline{a^3+b^3+c^3+d^3} + \&c. \times A - \overline{a^2+b^2+c^2+d^2}$$
$$+\ \&c. \times B + \overline{a+b+c+d} + \&c. \times C - 4D.$$

End of the SECOND PART.

A
TREATISE
OF
ALGEBRA.

PART III.
Of the Application of *Algebra* and *Geometry* to each other.

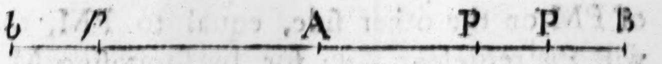
CHAP. I.
Of the Relation between the equations of Curve Lines and the figure of thoſe Curves, in general.

§ 1. IN the two firſt parts we conſidered Algebra as independent of Geometry; and demonſtrated its operations from its own principles. It remains that we now explain the uſe of Algebra in the reſolution of geometrical problems;

or reaſoning about geometrical figures; and the uſe of geometrical lines and figures in the reſolution of equations. The mutual intercourſe of theſe ſciences has produced many extenſive and beautiful Theories, the chief of which we ſhall endeavour to explain, beginning with the relation betwixt curve lines and their equations.

§ 2. We are now to conſider quantities as repreſented by *lines*; a *known* quantity by a *given* line, and an *unknown* by an *undetermined* line.

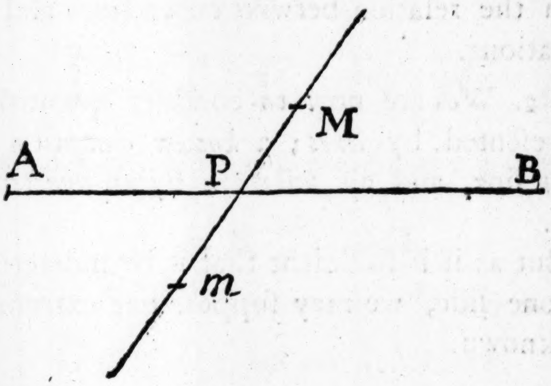
But as it is ſufficient that it be indetermined on one ſide, we may ſuppoſe one extremity to be known.



Thus the line AB, whoſe extremities A and B are both determined, may repreſent a given quantity: while AP, whoſe extremity P is undetermined, may repreſent an undetermined quantity. A leſſer undetermined quantity may be repreſented by AP, taking P nearer to A; and, if you ſuppoſe P to move towards A, then will AP, ſucceſſively, repreſent all quantities leſs than the firſt AP; and after P has coincided with A, if it proceed in the ſame direction to the place *p*, then will A*p* repreſent a negative quantity, if AP was ſuppoſed poſitive.

If AP repreſent x, and $Ap = AP$, then will Ap repreſent $-x$; and for the ſame reaſon, if AB repreſent $(+a,)$ then will Ab $(= AB)$ repreſent $(-a)$.

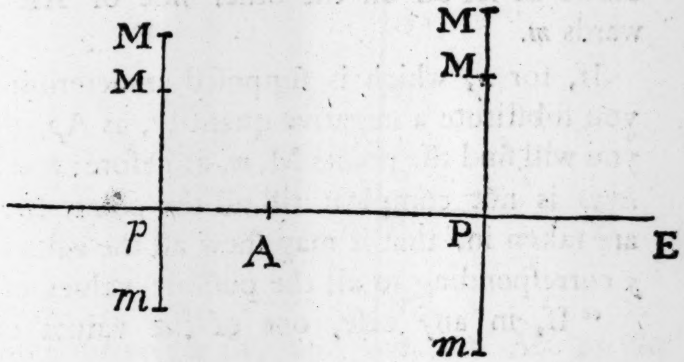
§ 3. After the ſame manner, if PM repreſent $+y$, and you take Pm, the continuation of PM on the other ſide, equal to PM, then will Pm repreſent $-y$: for, by ſuppoſing M to move towards P, the line PM decreaſes; when M comes to P, then PM vaniſhes; and after M has paſſed P, towards m, it becomes negative.



§ 4. In Algebra, the root of an equation, when it is an impoſſible quantity, has its expreſſion; but in Geometry, it has none. In Algebra you obtain a general reſolution, and there is an expreſſion, in all caſes, of the thing required; only, within certain bounds, that expreſſion repreſents an *imaginary* quantity, or rather, "*is the ſymbol of an operation which, in that caſe, cannot be performed*;" and ſerves only to ſhew the *geneſis* of the quantity, and the *limits* within which it is poſſible.

In the geometrical reſolution of a queſtion, the thing required is exhibited only in thoſe caſes when the queſtion admits of a *real* ſolution; and, beyond thoſe limits, no ſolution appears. So in finding the interſections of a given circle and a ſtraight line, if you determine them by an equation, you will find two general expreſſions for the diſtances of the points of interſection from the perpendicular drawn from the center on the given line. But, geometrically, thoſe interſections will be exhibited only when the diſtance of the ſtraight line from the center is leſs than the radius of the given circle.

§ 5. "When in an equation there are two undetermined quantities, x and y, then for each particular value of x, there may be as many values of y as it has dimenſions in that equation."



So that, if AP (a part of the indefinite line AE) repreſent x, and the perpendiculars PM repreſent the correſponding values of y, then there will be as many points (M,) the extremities of theſe perpendiculars or *ordinates*, as there are dimenſions of y in the equation. And the values of PM will be the roots of the equation ariſing by ſubſtituting for x its particular value AP in any caſe.

From which it appears, how, when an equation is given, you may determine as many of the points M as you pleaſe, and draw the line that ſhall paſs through all theſe points; "which is called the *locus* of the equation."

§ 6. When any equation involving two unknown quantities (x and y) is propoſed, then ſubſtituting for x any particular value AP, if the equation that ariſes has all its roots poſitive, the points M will lie on one ſide of AE; but if any of them are found negative, then theſe are to be ſet off on the other ſide of AE towards m.

If, for x, which is ſuppoſed undetermined, you ſubſtitute a negative quantity, as Ap, then you will find the points M, m, as before: and the *locus* is not compleat till all the points M, m, are taken in, that it may ſhew all the values of y correſponding to all the poſſible values of x.

" If, in any caſe, one of the values of y

vaniſh, then the point M coincides with P, and the *locus* meets with AE in that point."

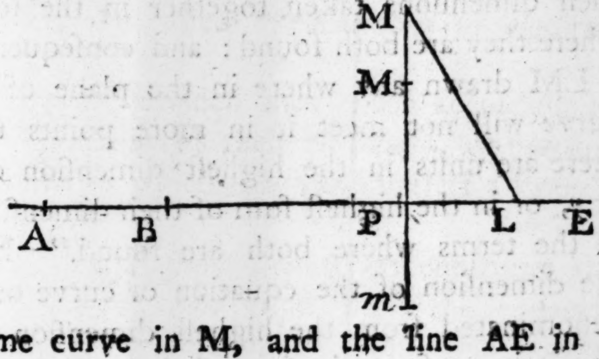
"If one of the values of y becomes infinite, then it ſhews that the curve has an *infinite arc*: and, in that caſe, the line PM becomes an *aſymptote* to the curve, or touches it at an infinite diſtance," if AP is itſelf finite.

"If, when x is ſuppoſed infinitely great, a value of y vaniſh, then the curve approaches to AE produced as an aſymptote."

"If any values of y become *impoſſible*, then ſo many points M vaniſh."

§ 7. From what has been ſaid it appears, that when an equation is propoſed involving two undetermined quantities (x and y) "there may be as many interſections of the curve that is the *locus* of the equation, and of the line PM as there are dimenſions of y in the equation; and as many interſections of the curve and the line AE as there are dimenſions of x in the equation."

If you draw any other line LM meeting the



ſame curve in M, and the line AE in the

given angle ALM. Suppoſe $LM = u$, and $AL = z$; "then the equation involving u and z, ſhall not riſe to more dimenſions then y and x had in the propoſed equation, or, than the ſum of their dimenſions in any of its terms."

For, ſince the angles PLM, MPL, PML, are given, it follows that, the ſines of theſe angles being ſuppoſed to one another as l, m, n, $PM : ML \ (y : u) :: l : m$; and conſequently $y = \frac{lu}{m}$: and that $PL : ML :: n : m$; ſo that

$PL = \frac{nu}{m}$, and $x = AP \ (= AL - PL) = z - \frac{nu}{m}$. Subſtitute, for y and x, in the propoſed equation theſe values $\frac{lu}{m}$ and $z - \frac{nu}{m}$, and it is obvious (ſince u and z are of one dimenſion only in the values of y and x) that in the equation which will ariſe, z and u will not have more dimenſions than the higheſt dimenſion of x and y in the propoſed equation, or the higheſt ſum of their dimenſions taken together in the terms where they are both found: and conſequently, "LM drawn any where in the plane of the curve will not meet it in more points than there are units in the higheſt dimenſion of x or y, or in the higheſt ſum of their dimenſions, in the terms where both are found." Now the dimenſion of the equation or curve being denominated from the higheſt dimenſion of x or y in it, or from the ſum of their dimenſions where they are moſt; we conclude, that "the number of points in which the curve can meet with any ſtraight line, is equal to the number that expreſſes the dimenſion of the curve."

It appears alſo from this article, how, when an equation of a curve is given expreſſing the relation of the ordinate PM and abſciſſe AP, you may transform it, ſo as to expreſs the relation between any other ordinate ML and the abſciſſe AL, by ſubſtituting for y its value $\frac{lu}{m}$, and for x its value $z - \frac{nu}{m}$.

Or, if you would have the abſciſſe begin at any other point B, ſuppoſing $AB = e$, ſubſtitute for x not $z - \frac{nu}{m}$, but $z - \frac{nu}{m} + e$.

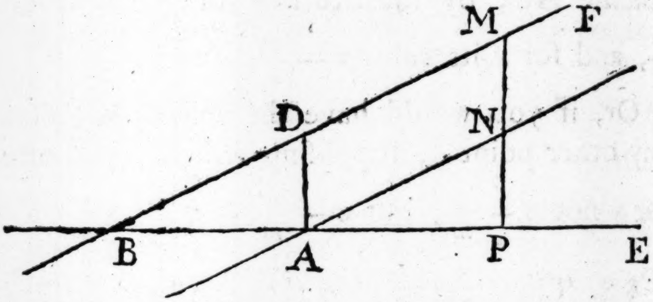
§ 8. Thoſe curve lines that can be deſcribed by the reſolution of equations, the relation of whoſe ordinates PM and abſciſſes AP can be expreſſed by an equation involving nothing but determined quantities beſides theſe ordinates and abſciſſes, are called "*geometrical* or *algebraic* curves."

They are divided into *orders* according to the dimenſions of their equations, or number of points in which they can interſect a ſtraight line.

The *ſtraight lines* themſelves conſtitute the *firſt* order of lines; and when the equation expreſſing the relation of x and y is of one dimenſion only, the points M muſt be all found in a ſtraight line conſtituting a given angle with AE.

Suppoſe, for example, that the equation given is $ay - bx - cd = 0$, and that the *locus* is required.

Since $y = \frac{bx + cd}{a}$, it follows, that, APM being a right angle, if you draw AN making the angle NAP ſuch that its coſine be to its ſine as a to b; and drawing AD parallel to the ordinates PM, and equal to $\frac{cd}{a}$, through D you draw DF parallel to AN, DF will be the locus required. Where you are to take AD on the ſame ſide of the line AE, with PN, if bx and cd have the ſame ſign, but on the contrary ſide of AE if they have contrary ſigns.



§ 9. Thoſe curves whoſe equations are of *two* dimenſions conſtitute the *ſecond order* of lines, and the *firſt kind* of *curves*. Their interſections with a ſtraight line can never exceed two, by § 7.

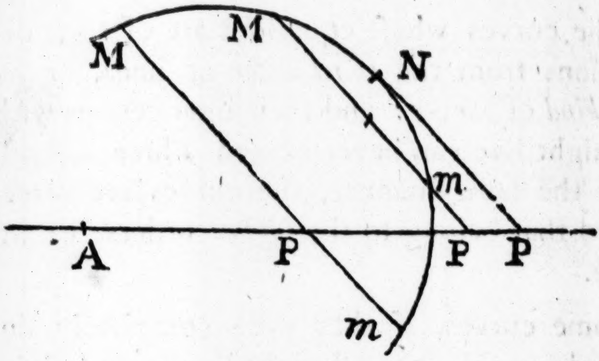
The curves whoſe equations are of *three* dimenſions from the *third order* of lines, or *ſecond kind* of *curves:* and their interſections with a ſtraight line can never exceed Three. And, after the ſame manner, the curves are determined that belong to the *higher* orders, to infinity.

Some curves, if they were completely deſcribed, could cut a ſtraight line in an infinite number of points; but theſe belong to none of the orders we have mentioned; they are not geometrical or algebraic curves, for the relation betwixt their ordinates and abſciſſes cannot be expreſſed by a finite equation involving only ordinates and abſciſſes with determined quantities.

§ 10. As "the roots of an equation become impoſſible always in pairs, ſo the interſections of the curve and its ordinate PM muſt vaniſh in pairs," if any of them vaniſh.

Let PM cut the curve in the points M and *m*, and by moving parallel to itſelf come to touch it in the point N; then the two points of interſection, M and *m*, go into one point of contact N. If PM ſtill move on parallel to itſelf, the points of interſection will, beyond

N, become imaginary; as the two roots of



an equation firſt become equal and then imaginary.

§ 11. The curves of the 3d, 5th, 7th orders, and all whoſe dimenſions are odd numbers, muſt have, at leaſt, two infinite arcs; ſince equations whoſe dimenſions are odd numbers have always one real root *at leaſt*; and conſequently, for every value of *x*, the equation by which *y* is determined muſt, at leaſt, have one real root: ſo that as *x* (or AP may be increaſed *in infinitum* on both ſides, it follows that M muſt go off *in infinitum* on both ſides, without limit.

Whereas, in the curves whoſe dimenſions are even numbers, as the roots of their equations may become all impoſſible, it follows that the figure of the curve may be like a *circle* or *oval* that is limited within certain bounds, beyond which it cannot extend.

§ 12. When two roots of the equation by which y is determined become equal, either "the ordinate PM touches the curve," two points of interſection, in that caſe, going into a point of contact; or, "the point M is a *punctum duplex* in the curve;" two of its arcs interſecting each other there: or, "ſome oval that belongs to that kind of curve becoming infinitely little in M, it vaniſhes into what is called a *punctum conjugatum.*"

If, in the equation, y be ſuppoſed $= 0$, then "the roots of the equation by which x is determined, will give the diſtances of the points where the curve meets AE from A." And, if two of thoſe roots be found equal, then either "the curve *touches* the line AE;" or, "AE paſſes through a *punctum duplex* in the curve." When y is ſuppoſed $= 0$, if one of the values of x vaniſh, the curve, in that caſe, paſſes through A." If two vaniſh, then either "AE touches the curve in A;" or, "A is a *punctum duplex.*"

"As a *punctum duplex* is determined from the equality of two roots, ſo is a *punctum triplex* determined from the equality of *three* roots.

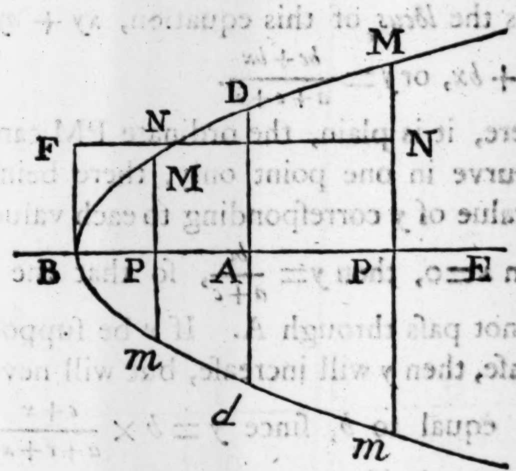
§ 13. A few examples will make theſe obſervations very plain. Suppoſe it is required to deſcribe the line that is the *locus* of this equation, $y^2 = ax + ab$, or $y^2 - ax - ab = 0$. Since $y = \pm\sqrt{ax+ab}$, and ſince *a* and *b*, are given in variable quantities, if you aſſume AP ($= x$) of a known value, it will be eaſy to find $\sqrt{ax+ab}$; and ſetting off PM on one ſide equal to $\sqrt{ax+ab}$, and P*m* on the other equal to PM, the points M and *m* will belong to the locus required. And for every poſitive value of AP you will thus obtain a point of the locus on each ſide. The greater AP ($= x$) is taken, the greater does the $\sqrt{ax+ab}$ become, and conſequently PM and P*m* become the greater.

If AP be ſuppoſed infinitely great, PM and P*m* will alſo become infinitely great; and conſequently the locus has two infinite arcs that go off to an infinite diſtance from AE and from AD. If you ſuppoſe *x* to vaniſh, $y = \pm\sqrt{ab}$; ſo that *y* does not vaniſh in that caſe but paſſes through D and *d*, taking AD and A*d* $= \sqrt{ab}$ a mean proportional betwixt *a* and *b*.

If you now ſuppoſe that the point P moves to the other ſide of A, then you muſt, in the equation, ſuppoſe *x* to become negative, and $y = \pm\sqrt{ab - ax}$; ſo that *y* will have two values as before, while *x* is leſs than *b*. But if AB$= b$, and you ſuppoſe the point P to come to B, then $ab = ax$, and $y = \pm\sqrt{ab - ax} = 0$. That is, PM and P*m* vaniſh; and the curve there meets the line AE. If you ſuppoſe P to move from

A

A beyond B, than x becomes greater than b, and ax greater than ab, ſo that $ab-ax$ being



negative, $\sqrt{ab-ax}$ becomes imaginary, and the two values of y become imaginary; that is, beyond B there are no ordinates that meet the curve, and conſequently, on that ſide, the curve is limited in B.

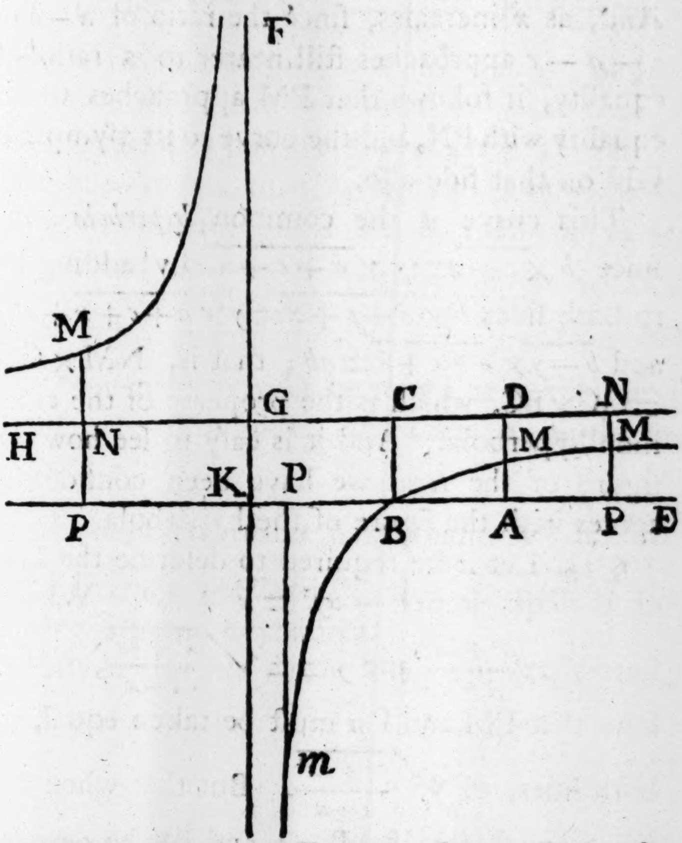
All this agrees very well with what is known by other methods, that the curve whoſe equation is $y^2=ax+ab$, is a parabola whoſe vertex is B, axis BE, and parameter equal to a. For ſince $BP=b\pm x$, and $PM=y$, if BF be equal to a; then the rectangle BN $(=ab\pm ax)$ will be equal to PMq $(=y^2;)$ which is the known property of the *parabola*. And it is obvious, that the figure of the parabola is ſuch as we have determined this locus to be from the conſideration of its equation.

§ 14. Let it be required to deſcribe the line that is the *locus* of this equation, $xy + ay + cy = bc + bx$, or $y = \frac{bc+bx}{a+c+x}$.

Here, it is plain, the ordinate PM can meet the curve in one point only, there being but one value of y correſponding to each value of x. When $x=0$, then $y = \frac{bc}{a+c}$, ſo that the curve does not paſs through A. If x be ſuppoſed to increaſe, then y will increaſe, but will never become equal to b, ſince $y = b \times \frac{c+x}{a+c+x}$, and $a + c + x$ is always greater than $c + x$. If x be ſuppoſed infinite, then the terms a and c vaniſh compared with x, and conſequently $y = b \times \frac{x}{x} = b$; from which it appears, that taking AD $= b$; and drawing GD parallel to AE, it will be an *aſymptote*, and touch the curve at an infinite diſtance.

If x be now ſuppoſed negative, and AP be taken on the other ſide of A, then ſhall $y = b \times \frac{c-x}{a+c-x}$; and if x be taken, on that ſide, $= c$, then ſhall $y = b \times \frac{c-c}{a} = 0$; ſo that the curve muſt paſs through B, if AB $= c$.

If x be ſuppoſed greater than c, then will $c-x$ become negative, and the ordinate will become negative and lie on the other ſide of AE, till x becomes equal to $a+c$, and then $y = b \times \frac{-a}{0}$,



or *infinite*; ſo that if AK be taken $= a + c$, the ordinate KL will be an *aſymptote* to the curve.

If x be taken greater than $a+c$, or AP greater than AK, then both $c-x$ and $a+c-x$ become negative; and conſequently $y \left(= b \times \frac{x-c}{x-a-c}\right)$ becomes poſitive; and ſince $x-c$ is always greater than $x-a-c$, it follows that y will be always greater than b or KG, and conſequently the reſt of the curve lies in the angle FGH. And, as x increaſes, ſince the ratio of $x-c$ to $x-a-c$ approaches ſtill nearer to a ratio of equality, it follows that PM approaches to an equality with PN, and the curve to its aſymptote GH on that ſide alſo.

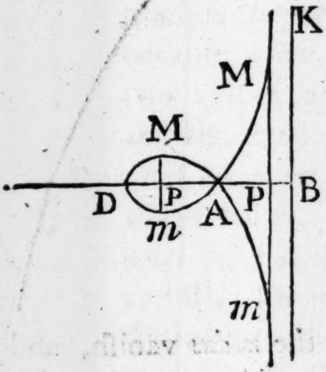
This curve is the common *hyperbola*; for ſince $b\times\overline{c+x}=y\times\overline{a+c+x}$, by adding ab to both ſides $b\times\overline{a+c+x}=y\times\overline{a+c+x}+ab$; and $\overline{b-y}\times\overline{a+c+x}=ab$; that is, $\mathrm{NM}\times\mathrm{GN}=\mathrm{GC}\times\mathrm{BC}$, which is the property of the common hyperbola. And it is eaſy to ſee how the figure of the *locus* we have been conſidering agrees with the figure of the hyperbola.

§ 15. Let it be required to deſcribe the *locus* of the equation $cy^2-xy^2=x^3+bx^2$. Where, ſince $y^2=\frac{x^3+bx^2}{c-x}$ and $y=\pm\sqrt{\frac{x^3+bx^2}{c-x}}$, it follows that PM and Pm muſt be taken equal, on both ſides, to $\sqrt{\frac{x^3+bx^2}{c-x}}$. But that when x is taken equal to c, if $\mathrm{AB}=c$, and BK be perpendicular to AB, then BK muſt be an *aſymptote* to the curve. If x be ſuppoſed greater than c, or AP greater than AB, then $c-x$ being negative, the fraction $\frac{x^3+bx^2}{c-x}$ will become negative, and its ſquare root impoſſible. So that no part of the *locus* can be found beyond B. If x be ſuppoſed negative, or P taken on the other ſide of A, then $y = \pm\sqrt{\frac{-x^3+bx^2}{c+x}}$, the ſign of x^3 and x being changed, but not the ſign of bx^2; becauſe the ſquare of a negative is the ſame as the ſquare of a poſitive, but its cube is negative: while x is leſs than b, the values of y will be real and equal; but if $x = b$, then the values of y vaniſh, becauſe, in that caſe,

$$y = \pm\sqrt{\frac{-x^3+bx^2}{c-x}} = \sqrt{\frac{-b^3+b^3}{c-x}} = 0;$$ and conſequently, if AD be taken $= b$, the curve will paſs through D, and there touch the ordinate.

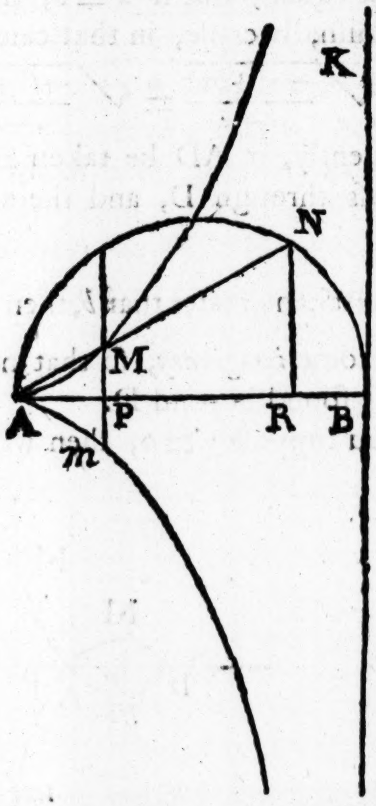
If x be taken greater than b, then $\pm\sqrt{\frac{-x^3+bx^2}{c+x}}$ will become *imaginary*, ſo that no part of the curve is found beyond D.

If you ſuppoſe $y = 0$, then will $x^3 + bx^2 = 0$



be an equation whoſe roots are $-b$, 0, 0, from which it appears that the curve paſſes twice through the point A, and has, in A, a *punctum duplex*. This *locus* is a line of the third order, BK is its aſymptote, and it has a *nodus* betwixt A and D.

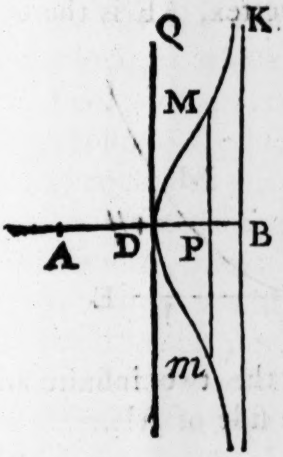
If you ſuppoſe b to vaniſh in the equation, ſo that $cy^2 - xy^2 = x^3$, then will A and D coincide, and the *nodus* vaniſh, and the curve will have in the point A a *cuſpis*, the two arcs AM and



and A*m* touching one another in that point. And this is the ſame curve which by the ancients was called the *ciſſoid* of *Diocles*, the line AB being the diameter of the generating circle, and BK the aſymptote.

For, if BR be equal to AP, and the ordinate RN be raiſed meeting the circle in N, and AN be drawn, it will cut the perpendicular PM in M a point of the ciſſoid. So that if M be a point in the ciſſoid, $AP : PM :: AR : RN :: \sqrt{AR} : \sqrt{BR} :: \sqrt{BP} : \sqrt{AP}$, and conſequently, $BP \times PMq = AP$ cub. that is, $\overline{c-x} \times y^2 = x^3$: which is the equation the *locus* of which was required.

If, inſtead of ſuppoſing b poſitive, or equal to nothing, we now ſuppoſe it negative, the equation will be $cy^2 - xy^2 = x^3 - bx^2$, the curve will paſs through D, as before, and taking $AB = c$, BK will be its aſymptote: it will have a *punctum conjugatum* in A, becauſe when y vaniſhes, two values of x vaniſh, and the third becomes equal to b or AD. The whole curve, beſides this *point* A, lies between DQ and BK. Theſe are demonſtrated after the ſame manner as in the firſt caſe.

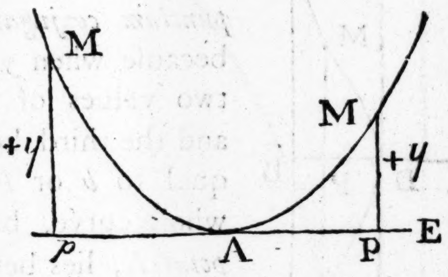


§ 16.

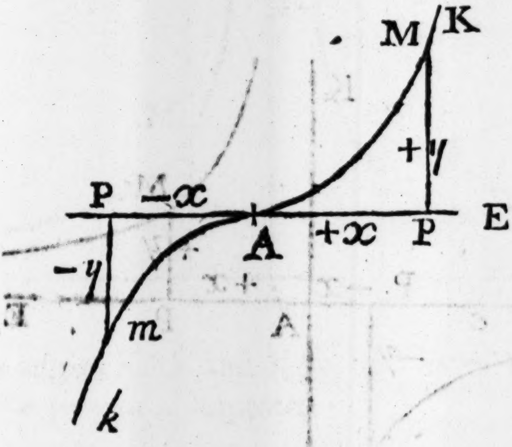
§ 16. If an equation is propoſed, as $y = ax^n + bx^{n-1} + cx^{n-2}$, *&c.* and n is an even number, then will the *locus* of the equation have two infinite arcs lying on the ſame ſide of AE. For, if x become infinite, whether poſitive or negative, x^n will be poſitive, and ax^n have the ſame ſign in either caſe; and as ax^n becomes infinitely greater than the other terms bx^{n-1}, cx^{n-2}, *&c.* it follows that the infinite values of y will have the ſame ſign in theſe caſes; and conſequently, the two infinite arcs of the curve will lie on the ſame ſide of AE.

But if n be an odd number, then when x is negative, x^n will be negative, and ax^n will have the contrary ſign to what it has when x is poſitive; and therefore the two infinite arcs, in this caſe, will lie on different ſides of AE, and tend towards parts directly oppoſite.

Thus the locus of the equation $ay = x^2$ is the parabola. A is the vertex, AE is the tangent at the vertex; and the two infinite arcs lie manifeſtly on the ſame ſide of AE.



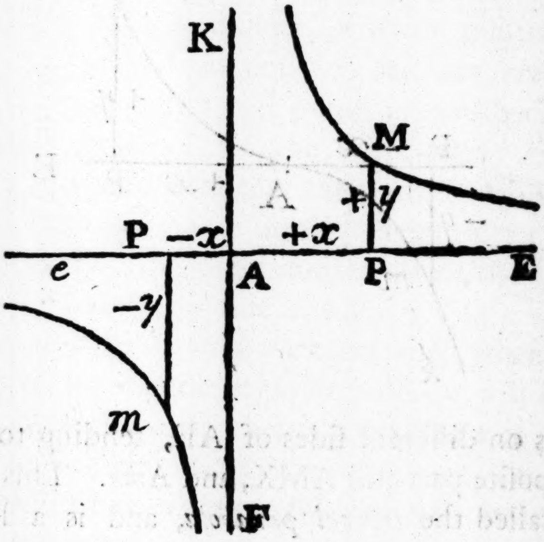
But the *locus* oſ the equation $a^2y=x^3$, where the index of x is an odd number, has its two



arcs on different ſides of AE, tending towards oppoſite parts, as AMK, and A*mk*. This curve is called the *cubical parabola*, and is a line of the third order.

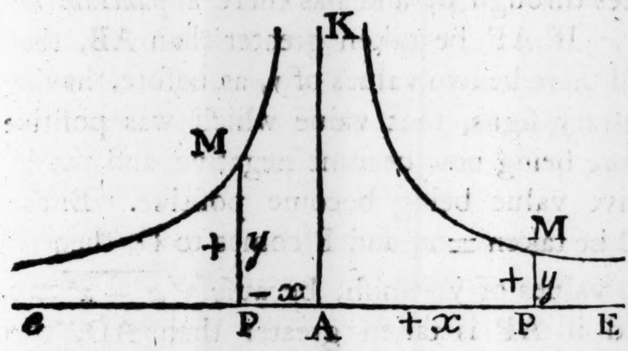
The *locus* of the equation $a^3y=x^4$ is of a figure like the common parabola; and "all thoſe loci, in whoſe equations y is of one dimenſion, x of an *even* number of dimenſions: But thoſe *loci* are like the cubical parabola, in whoſe equations y is of one dimenſion only, and x of an *odd* number of dimenſions." And this Rule is even true of the locus of the equation $y=x$, which is a ſtraight line cutting AE in an angle of 45°; which manifeſtly goes off as the cubical parabola does to infinity, towards oppoſite parts, and on different ſides of AE.

§ 17. If the *locus* of the equation $yx^n = a^{n+1}$ is required;



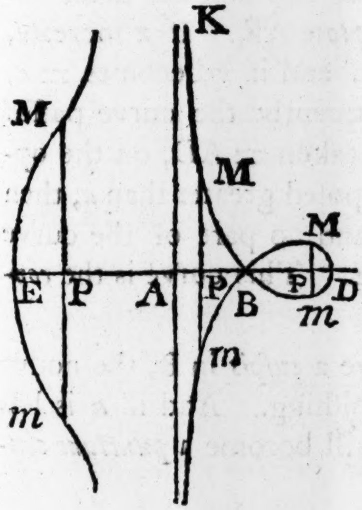
If n is an *odd* number, then when x is poſitive, $y = \frac{a^{n+1}}{x^n}$; but when x is negative, then $y = -\frac{a^{n+1}}{x^n}$; ſo that this curve muſt all lie in the vertically *oppoſite* angles KAE, FAE, (as the common *hyperbola*:) FK, E*e*, being aſymptotes.

But if n is an *even* number, then y is always poſitive, whether x be poſitive or negative, becauſe x^n, in this caſe, is always poſitive; and therefore the curve muſt all lie in the two *ad-*



jacent angles KAE and KA*e*, and have AK and AE for its two aſymptotes.

§ 18. Let the equation given be $\overline{a^2-x^2}\times\overline{x-b}|^2=x^2y^2$; ſo that $y=\pm\sqrt{a^2-x^2}\times\frac{x-b}{x}$.



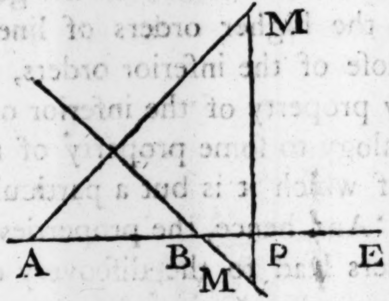
If $x=0$, then y becomes infinite, and therefore the ordinate at A is an *aſymptote* to the curve. If AB $=b$, and P be taken betwixt A and B, then ſhall PM and P*m* be equal, and lie on different ſides of the abſciſſe AP. If $x=b$, then the two values of y vaniſh, becauſe $x-b=0$; and conſequently, the curve paſſes through B, and has there a *punctum duplex.* If AP be taken greater than AB, then ſhall there be two values of y, as before, having contrary ſigns, that value which was poſitive before being now become negative, and the negative value being become poſitive. But if AD be taken $= a$, and P comes to D, then the two values of y vaniſh, becauſe $\sqrt{a^2-x^2}=0$. And if AP is taken greater than AD, then a^2-x^2 becomes negative, and the value of y *impoſſible:* and therefore, the curve does not go beyond D.

If x now be ſuppoſed negative, we ſhall find $y=\pm\sqrt{a^2-x^2}\times\overline{b+x}\div x$. If x vaniſh, both theſe values of y become infinite, and conſequently, the curve has two infinite arcs, on each ſide of the *aſymptote* AK. If x increaſe, it is plain y diminiſhes, and if x becomes $= a$, y vaniſhes, and conſequently the curve paſſes through E, if AE be taken $=$ AD, on the oppoſite ſide. If x be ſuppoſed greater than a, then y becomes *impoſſible*; and no part of the curve can be found beyond E. This curve is the *conchoid* of the ancients.

If $a=b$, it will have a *cuſpis* in B, the nodus betwixt B and D vaniſhing. And if a is leſs than b, the point B will become a *punctum conjugatum.*

From what has been ſaid an error may be corrected of an Author in the *Memoires de l'Acad. Royale des Sciences*, who gives this curve no infinite arcs, but only a *double nodus*. Some other errors of the ſame kind may be corrected in that Treatiſe, from what we have ſaid.

§ 19. If the propoſed equation can be reſolved into two equations of lower dimenſions, without affecting either y or x with any radical ſign, then the *locus* ſhall conſiſt of the two *loci* of thoſe inferior equations. Thus the locus of the equation $y^2 - 2xy + by + x^2 - bx = 0$ is found to be two ſtraight lines cutting the abſciſſe AE in angles of 45°, in the points A and B, whoſe diſtance AB $= b$, becauſe that equation is reſolved into theſe two $y - x = 0$, and $y - x + b = 0$.



After the ſame manner, ſome *cubic* equations can be reſolved into three ſimple equations, and then the *locus* is *three* ſtraight lines; or may be reſolved into a *quadratic* and *ſimple* equation, and then the *locus* is a *conic ſection* and a *ſtraight line*.

In general, "the curves of the ſuperior orders include all the curves of the inferior orders; and whatever is demonſtrated generally of any one order, is alſo true of the inferior orders." So, for example, any general property of the conic ſections hold true of two ſtraight lines as well as of a conic ſection. Particularly that "the rectangles of the ſegments of parallels bounded by them, will be always to one another in a given ratio." The general properties of the lines of the third *order* are true of three ſtraight lines, or of any one ſtraight line and a conic ſection. And, as the general properties of the higher orders of lines deſcend alſo to thoſe of the inferior orders, ſo there is ſcarce any property of the inferior orders, but has an analogy to ſome property of the higher orders; of which it is but a particular caſe or inſtance. And hence, the properties of the inferior orders lead to the diſcovery of thoſe of the ſuperior orders *.

§ 20. We have ſhewed how to judge of the figure of a *locus* from the conſideration of its equation. And when a *locus* is to be deſcribed exactly, for every value of x you muſt, by the reſolution of equations, according to the Rules in *Part* II. find the correſponding values of y, and determine from theſe values the points of the *locus*.

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* See the Appendix.

But there are geometrical *conſtructions* by which the roots of equations can be determined more commodiouſly for this purpoſe. And, as by theſe conſtructions we deſcribe the *loci* of the equations, ſo reciprocally when *loci* are deſcribed, they are uſeful in determining the roots of equations; both which ſhall be explained in the following *Chapter*. Then we ſhall give an account of the moſt general and ſimple methods of deſcribing theſe *loci* by the mechanical motion of angles and lines, whoſe interſections trace the curve; or of conſtructing them by finding geometrically any number of their points.

CHAP. II.

Of the Conſtruction of Quadratic Equations; and of the Properties of the Lines of the ſecond order.

§ 21. THE general equation expreſſing the nature of the lines of the ſecond order, having all its terms and coefficients, will be of this form;

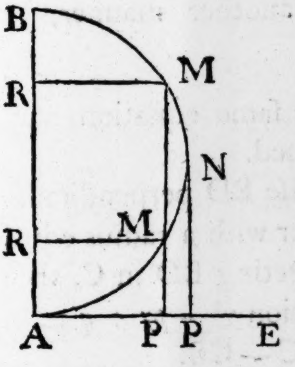
$$\left.\begin{array}{rrr} y^2 & + axy & + cx^2 \\ & + by & + dx \\ & & + e \end{array}\right\} = 0.$$

Where a, b, c, d, e, repreſent any given quantities with their proper ſigns prefixed to them.

If a quadratic equation is given, as $y^2 + py + q = 0$, and, by comparing it with the preceding, if you take the quantities a, b, c, d, e, and x ſuch that $ax + b = p$, and $cx^2 + dx + e = q$, then will the values of y in the firſt equation be equal to the values of it in the ſecond; and if the *locus* be deſcribed belonging to the firſt equation, the two values of the ordinate when $ax + b = p$ and $cx^2 + dx + e = q$, will be the two roots of the equation $y^2 + py + q = 0$.

And as *four* of the given quantities a, b, c, d, e, may be taken at pleaſure, and the *fifth*, with the abſciſſe x, determined, ſo that $ax + b$ may be ſtill equal to p, and $cx^2 + dx + e = q$; hence there are innumerable ways of conſtructing the ſame equation. But thoſe *loci* are to be preferred which are deſcribed moſt eaſily; and therefore, the *circle*, of all conic ſections, is to be preferred for the reſolution of quadratic equations.

§ 22. Let AB be perpendicular to AE, and upon AB deſcribe the ſemicircle BMMA. If AP be ſuppoſed equal to x, $AB = a$, and $PM = y$, then making MR, MR, perpendiculars to the diameter AB, ſince $AR \times RB = RMq$, and $AR = y$, $RB = a - y$, $RM = x$, it follows that $\overline{a - y} \times y = x^2$, and $y^2 - ay + x^2 = 0$. And, if an equation $y^2 - py + q = 0$, be propoſed to be reſolved, its roots will be the ordinate to the circle, PM and PM, to its tangent AE, if $a = p$, and $x^2 = q$: becauſe then the equation of the circle $y^2 - ay + x^2 = 0$, will be changed into the propoſed equation $y^2 - py + q = 0$.



We have therefore this conſtruction for finding the roots of the quadratic equation $y^2 - py + q = 0$; take $AB = p$, and on AB deſcribe a ſemicircle; then raiſe AE perpendicular to AB, and on it take AP $= \sqrt{q}$, that is, a mean proportional between 1 and q (*by* 13 *El.* 6.) then draw PM parallel to AB, meeting the ſemicircle in M, M, and the lines PM, PM ſhall be the roots of the propoſed equation.

It appears from the conſtruction that if $q = \frac{p^2}{4}$, or $\sqrt{q} = \frac{1}{2}p$, then AP $= \frac{1}{2}$AB, and the ordinate PN touches the curve in N, the two roots PM, PM, in that caſe, becoming equal to one another and PN.

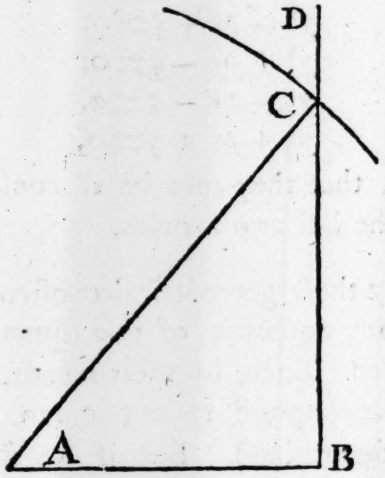
If AP be taken greater than $\frac{1}{2}$AB, that is, when $\sqrt{q}$ is greater than $\frac{1}{2}p$, or q greater than $\frac{1}{4}p^2$, the ordinates do not meet the circle, and the roots of the equation become *imaginary:* as we demonſtrated, in another manner, in *Part* II.

§ 23. The roots of the ſame equation may be otherwiſe thus determined.

Take AB $= \sqrt{q}$, and raiſe BD perpendicular to AB; from A as a center with a radius equal to $\frac{1}{2}p$, deſcribe a circle meeting BD in C, then the two roots of the equation $y^2 - py + q = 0$, ſhall be AC + CB, and AC — CB.

For theſe roots are $\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 - q}$, and $\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 - q}$; and AC $= \frac{1}{2}p$, CB $= \sqrt{AC^2 - CB^2}$ $= \sqrt{\frac{1}{4}p^2 - q}$, and conſequently theſe roots are AC ± CB.

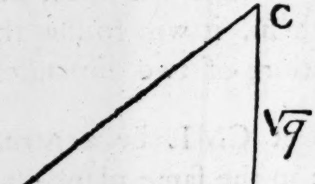
The roots of the equation $y^2 + py + q = 0$ are



$-AC \pm CB$; as is demonſtrated in the ſame manner.

§ 24. The roots of the equation $y^2 - py - q = 0$ are determined by this conſtruction.

Take $AB = \frac{1}{2}p$, $BC = \sqrt{q}$, draw AC; and the two roots ſhall be $AB \pm AC$. If the ſecond term is poſitive, then the roots ſhall be $-AB \pm AC$.



And all quadratic equations being reducible to theſe four forms,

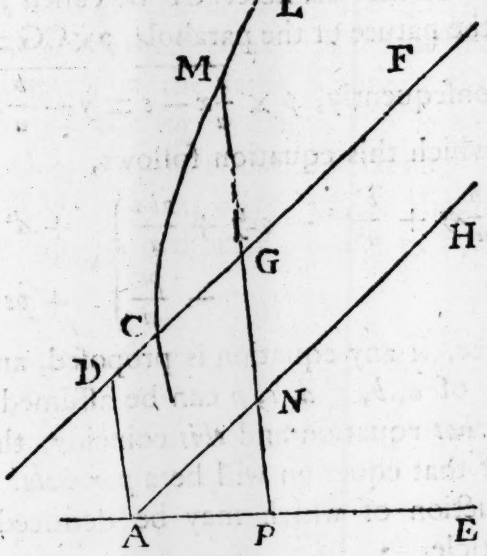
$$y^2 - py + q = 0,$$
$$y^2 + py - q = 0,$$
$$y^2 - py - q = 0,$$
$$y^2 + py + q = 0,$$

it follows, that they may be all conſtructed by this and the laſt two articles.

§ 25. By theſe geometrical conſtructions, the locus of any equation of two dimenſions may be deſcribed; ſince, by their means, the values of y that correſpond to any given value of x may be determined. But if we demonſtrate that theſe *loci* are always *conic ſections*, then they may more eaſily be deſcribed by the methods that are already known for deſcribing theſe curves.

In order to prove this, we ſhall enquire what equations belong to the different *conic ſections*; and, as it will appear that there is no equation of two dimenſions but muſt belong to one or other of them, it will follow that they are *loci* of all equations of two dimenſions.

§ 26. Let CML be a *parabola*; AE any line drawn in the ſame plane; and let it be required to find the equation expreſſing the relation betwixt the ordinate PM forming any given angle with AE, and the abſciſſe AP beginning at A any given point in the line AE.



Let CF be the diameter of the parabola whoſe ordinates are parallel to PM. Draw AH parallel to CF meeting PM in N; and AD parallel to PM meeting CF in D. Becauſe the angles HAE, APN, ANP, are given the lines AP, PN, AN, will be in a given ratio to each other: ſuppoſe them to be always as a, b, c; let $AD = d$, $DC = e$; and ſeeing AP $(= x)$: PN :: $a : b$, $PN = \frac{b}{a}x$; likewiſe AP : AN :: $a : c$, or $AN = \frac{c}{a}x$. And $GM = PM - PN - NG = y - \frac{b}{a}x - d$. But $CG = DG -$

DC

DC = AN − DC = $\frac{c}{a}x - e$. If now the *parameter* of the diameter CF be called p, then, from the nature of the parabola, $p \times \text{CG} = \text{GM}q$: and conſequently, $p \times \overline{\frac{c}{a}x - e} = \overline{y - \frac{b}{a}x - d}\,\Big|^2$, from which this equation follows,

$$y^2 - \frac{2b}{a}xy + \frac{b^2}{a^2}x^2 - 2dy + \left.\begin{array}{r} \frac{2bd}{a} \\ - \frac{pc}{a} \end{array}\right\} x \left.\begin{array}{r} + d^2 \\ + pe \end{array}\right\} = 0.$$

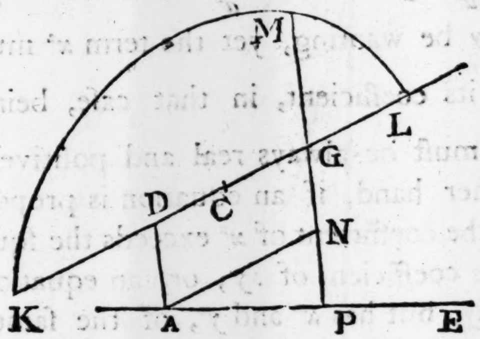
Whence, if any equation is propoſed, and ſuch values of a, b, c, d, e, p can be aſſumed as to make *that* equation and *this* coincide, then the *locus* of that equation will be a *parabola.* The conſtruction of which may be deduced from this article.

§ 27. In this general equation for the parabola, the coefficient of x^2 is the ſquare of half the coefficient of xy; and, "when any equation is propoſed that has this property, the *locus* of it is a *parabola.*" For, whatever coefficients affect the three laſt terms, they may be made to agree with the coefficients of the laſt terms of the general equation, by aſſuming proper values of p, c, and e.

It appears alſo, that "if the locus be a parabola, and the term xy be wanting, the term x^2 muſt alſo be wanting." And, "if any equation of two dimenſions be propoſed that wants both the terms, xy and x^2, it may be always accommodated to a *parabola.*

§ 28. The general equation for the *ellipſe* is deduced from the property of the ordinates of any diameter, in the ſame manner; the conſtruction of the figure being the ſame as in § 26. Only, in place of the *parabola,*

Let KML be an *ellipſe* whoſe diameter is KL, having its ordinates parallel to PM, and



let C be the center of the *ellipſe.* Suppoſe CL $= t$, and the *parameter* of that diameter $= p$, then GMq : CLq — CGq :: $p : 2t$. But, as in § 26, GM $= y - \frac{b}{a}x - d$, and CG $= \frac{c}{a}x - e$; therefore, $\overline{y - \frac{b}{a}x - d}\Big|^2 \times \frac{2t}{p} = t^2 - \frac{c^2}{a^2}x^2 + \frac{2ce}{a}x - e$: whence this equation;

$$y^2 - \frac{2b}{a}xy + \left.\begin{matrix} \frac{b^2}{a^2} \\ + \frac{pc^2}{2ta^2} \end{matrix}\right\} x^2 \quad - 2dy + \left.\begin{matrix} \frac{2bd}{a} \\ - \frac{pce}{at} \end{matrix}\right\} x \left.\begin{matrix} + d^2 \\ - \frac{pt}{2} \\ + \frac{pe^2}{2t} \end{matrix}\right\} = 0.$$

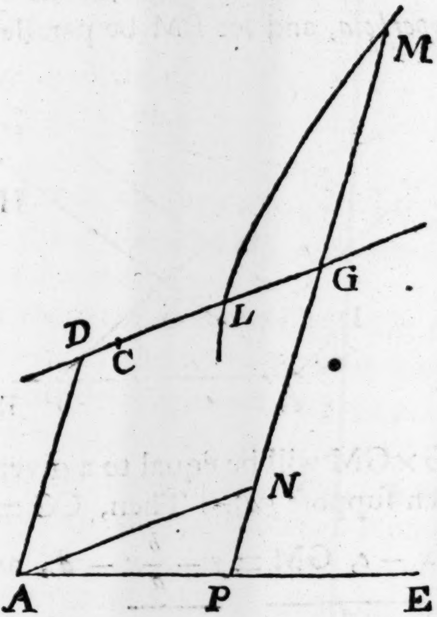
And if any equation is propoſed that can be made to agree with this general equation, by aſſuming proper values of a, b, c, d, p and e; then the *locus* of that equation will be an *ellipſe*.

§ 29. "In the general equation for the *ellipſe*, the terms x^2 and y^2 have the ſame ſign: and the coefficient of x^2 is always greater than the ſquare of half the coefficient of xy, becauſe $\frac{b^2}{a^2} + \frac{pc^2}{2ta^2}$ is greater than $\frac{b^2}{a^2}$. And although the term xy be wanting, yet the term x^2 muſt remain, its coefficient, in that caſe, being $\frac{p}{2t}$, which muſt be always real and poſitive. On the other hand, if an equation is propoſed in which the coefficient of x^2 exceeds the ſquare of half the coefficient of xy; or, an equation that wants xy, but has x^2 and y^2, of the ſame ſign, its *locus* muſt be an *ellipſe*."

§ 30. In the hyperbola, as $GMq : CGq - CLq :: p : 2t$; when t is a *firſt* diameter, the equation that ariſes will differ from the equation of the *ellipſe* only in the ſigns of the values of CGq and CLq, and conſequently will have this form,

$$\left.\begin{array}{llllll} y^2 - \frac{2b}{a}xy & + \frac{b^2}{a^2}x^2 & - 2dy & + \frac{2bd}{a}x & + d^2 \\ & - \frac{pc^2}{2ta^2}x^2 & & + \frac{pce}{at}x & + \frac{pt}{2} \\ & & & & - \frac{pe^2}{2t} \end{array}\right\} = 0.$$

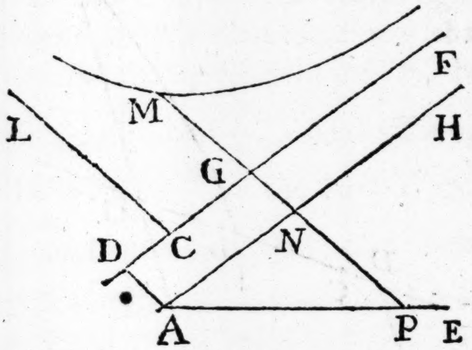
If t be a *ſecond* diameter, then $\frac{pt}{2}$ will be negative.



In this equation, it is manifeſt that the coefficient of the term x^2 is leſs than the ſquare of half the coefficient of xy; and, that when the term xy is wanting, the term x^2 muſt be negative. And, reciprocally, "if an equation is propoſed where the coefficient of x^2 is leſs than the ſquare of half the coefficient of xy; or where xy is wanting and y^2 and x^2 have contrary ſigns, the locus of that equation muſt be an *hyperbola*."

§ 31.

§ 31. The equation of the *hyperbola* when its ordinates PM are parallel to an aſymptote does not come under the general equation of the laſt article. Let CF and CL be the *aſymptotes* of the *hyperbola*, and let PM be parallel to CL,



Then CG×GM will be equal to a given rectangle (which ſuppoſe ga). Then, $CG = DG - DC = \frac{c}{a}x - e$, $GM = y - \frac{b}{a}x - d$; and conſequently $\overline{y - \frac{b}{a}x - d} \times \overline{\frac{c}{a}x - e} = g \times a$: whence this equation,,

$$xy - \frac{b}{a}x^2 + \left.\begin{matrix}\frac{eb}{c} \\ -d\end{matrix}\right\} x \; - \frac{ea}{c}y + \left.\begin{matrix}\frac{eda}{c} \\ -\frac{ga^2}{c}\end{matrix}\right\} = 0.$$

Where only one of the terms y^2, x^2, can be found with xy; and where xy will be found without either of theſe terms, if AE and AH coincide, that is, if AE is parallel to the aſymptote DF.

It appears from this, that "if an equation is propoſed that either has xy the only term of two dimenſions; or, has xy and either x^2 or y^2 beſides, but not both of them, the locus of the equation ſhall be an *hyperbola*, one of whoſe *aſymptotes* ſhall be parallel to y or x according as it is y^2 or x^2 that is wanting in the equation."

§ 32. From all theſe compared together, it follows, that "*the* locus *of any equation of two dimenſions is a* conic ſection."

For if the term xy is wanting in the equation, and but one of the terms y^2, x^2 is found in it, the *locus* ſhall be a *parabola*; by § 27.

If xy is wanting, and x^2, y^2, have the ſame ſign, then the *locus* is an *ellipſe*. § 29.—But, when they have different ſigns, it is an *hyperbola*. § 30.

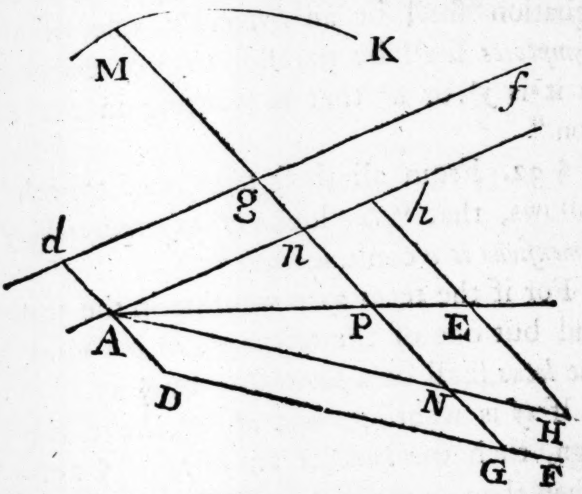
If xy is found in the equation, and x^2, y^2, are both wanting, or either of them, the *locus* is an *hyperbola*. § 31.

If both x^2 and y^2 are found in it, having contrary ſigns, the *locus* is ſtill an *hyperbola*.

If y^2 and x^2 have the ſame ſigns, then, according as the coefficient of x^2 is *greater*, *equal*, or *leſs* than the ſquare of half the coefficient of xy, the locus ſhall be an *ellipſe*, *parabola*, or *hyperbola*. § 27, 29, 30.

In any caſe therefore the *locus* of the equation is ſome *conic ſection*.

§ 33. Theſe may all be demonſtrated more directly from the conſideration of the general equation of the lines of the *ſecond order* in § 21. For it is obvious that, by § 25. *Part* II.



the ſecond term of that general equation may be exterminated by aſſuming $z = y + \frac{ax+b}{2}$, and it will be transformed into

$$\left.\begin{matrix} z^{2*} + \frac{4c-a^2}{4} \times x^2 \\ + \overline{d - \frac{ab}{2}} \times x \\ \qquad\qquad + e - \frac{b^2}{4} \end{matrix}\right\} = 0,$$

which, by tranſpoſing the laſt term, is,

$$z^2 = \overline{\frac{a^2}{4} - c} \times x^2 + \overline{\frac{ab}{2} - d} + x \left.\begin{matrix} + \frac{b^2}{4} \\ - e \end{matrix}\right\}.$$

Let MK be the locus of the equation: and if AH be drawn ſo that HE be to AE as $\frac{1}{2}a$ to unit, and AD, parallel to PM, be $= \frac{1}{2}b$, and through D the line DF be drawn parallel to AH, meeting PM in G, then ſhall GM $(= \text{PM} + \text{PN} + \text{NG} = y + \frac{1}{2}ax + \frac{1}{2}b) = z$. And if $\text{AH} = f$, then $\text{DG} = \text{AN} = fx$.

Suppoſe $\text{DG} = u$, and $x = \frac{u}{f}$. Inſtead of x ſubſtitute $\frac{u}{f}$, and the equation that reſults will expreſs the relation of GM and DG, of this form,

$$z^2 = \frac{a^2-4c}{4f^2} \times u^2 + \frac{ab-2d}{2f} \times u + \frac{1}{4}b^2 - e = 0.$$

Which will be an *hyperbola*, *parabola*, or *ellipſis*, according as the term $\frac{a^2-4c}{4f^2}$ is *poſitive*, *nothing*, or *negative*. That is, according as $\frac{a^2}{4}$ is *greater*, *equal* to, or *leſs* than c. But a was the coefficient of xy; from which it appears, that "the locus is an *ellipſe*, *parabola*, or *hyperbola*, according as the coefficient of x^2 is *greater*, *equal* to, or *leſs* than the ſquare of half the coefficient of xy."

It appears alſo, that "if the term xy be wanting, or $a = 0$, then the locus will be an *ellipſe*, *parabola*, or *hyperbola*, according as the term cx^2 is *poſitive*, *nothing*, or *negative*."

Hence likewiſe, if the term x^2 be wanting, and the term xy not wanting, then the term $\frac{a^2-4c}{4f^2}u^2$ being poſitive (becauſe $\frac{a^2}{4f^2}$ is always poſitive, whatever a or f be) "*the locus muſt be an* hyperbola."

Note, That part of the figure, on the other ſide of AE, which is marked with ſmall letters, anſwers to the caſe when the coefficient of y, in the general equation, *viz.* $ax+b$, is negative.

§ 24. The lines of the *ſecond order* have ſome general properties which may be demonſtrated from the conſideration of the general equation repreſenting them.

The general equation of § 21. by extermінating the ſecond term can be transformed into the equation,

$$z^2 = \frac{a^2-4c}{4f^2} \times u^2 + \frac{ab-2d}{2f} \times u + \frac{b^2}{4} - e.$$

From which we have

$$z = \pm\sqrt{\frac{a^2-4c}{4f^2} \times u^2 + \frac{ab-2d}{2f} + \frac{b^2}{4} - e}.$$

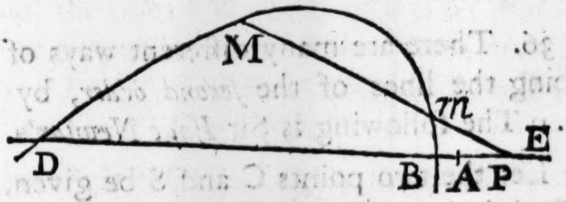
Where the two values of z are always equal, and have contrary ſigns, ſo that the line DF, on which the abſciſſes are taken, muſt biſect the ordinates, and conſequently, is a *diameter* of the conic ſection. And, as this has been demonſtrated generally, in any ſituation of the lines PM, it follows that if any parallels, as

M*m*, M*m*, be drawn meeting a conic ſection *, there is a line DF which can biſect all theſe parallels. And conſequently if any two parallels, M*m*, M*m*, are biſected in *G* and *g*, the line *Gg* that biſects theſe two, will biſect all the other lines parallel to them, terminated by the curve. "Which is a general property of all the conic ſections."

There is one caſe which muſt be excepted, when PM is parallel to an aſymptote, becauſe in that caſe it meets with the conic ſection only in one point.

§ 35. In the general equation of § 21, if you ſuppoſe $y=0$, there will remain $cx^2+dx+e=0$, by which the points are determined where the curve meets the abſciſſe AE.

Suppoſe it meets it in B and D, and that AB = A, and AD = B. Then ſhall −A and −B be the two roots of the equation $x^2+\frac{d}{c}x+\frac{e}{c}=0$; and therefore $\overline{x+A}\times\overline{x+B}=x^2+\frac{d}{c}x+\frac{e}{c}$: but $x+A=BP$, and $x+B=DP$;



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* Supply the figure.

therefore BP × DP $= x^2 + \frac{d}{c}x + \frac{e}{c}$. Now, it is manifeſt from the nature of equations, that if PM meet the curve in M and *m*, the rectangle of the roots PM and P*m* ſhall be equal to $cx^2 + dx + e$ the laſt term of the equation

$$\left.\begin{array}{r} y^2 + axy + cx^2 \\ + by + dx \\ + e \end{array}\right\} = 0.$$

We have therefore PM × P*m* $= cx^2 + dx + e$; and BP × DP $= x^2 + \frac{d}{c}x + \frac{e}{c}$; ſo that PM × P*m* : BP × DP $:: cx^2 + dx + e : x^2 + \frac{d}{c}x + \frac{e}{c} :: c : 1$. That is, "the rectangle of the ordinates PM, P*m* is to the rectangle of the ſegments of the abſciſſes, as, in a given ratio, *c* is to 1." Which is another general property of the lines of the *ſecond order*.

In a ſimilar manner the analogous properties of the lines of the higher orders are demonſtrated *.

§ 36. There are many different ways of deſcribing the lines of the *ſecond order*, by motion. The following is Sir *Iſaac Newton*'s.

† Let the two points C and S be given, and the ſtraight line AE in the ſame plane. Let the

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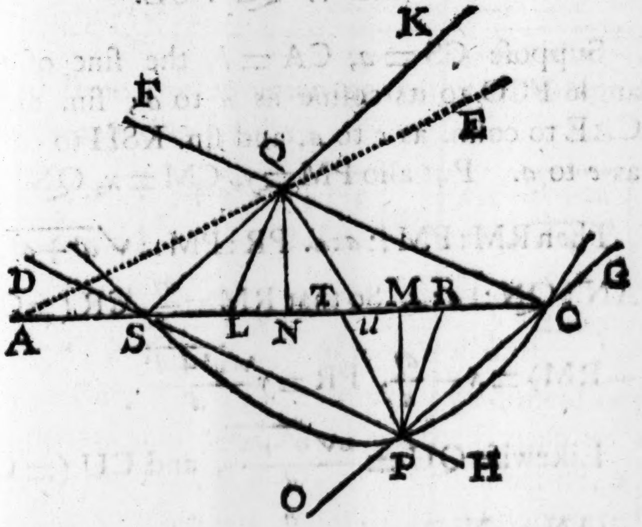
* See the APPENDIX.

† See *Geometria Organica*, Prop. I.

given angles FCG, KSH, revolve about the points C and S as poles, and let the interſection of the ſides CF, SK, be carried along the



ſtraight line AE, and the interſection of the ſides CO, SH, will deſcribe a line of the *ſecond order.*

Let the ſides CF, SK interſect each other in Q, and the ſides CO, SH, in P; let PM and QN be perpendicular on CS. Then draw PR, QU; PT, QL; ſo that $CUQ = CRP = FCG$; and $SLQ = STP = KSD$.

The angle $RCP = CQU$, ſince RCP makes two right ones with RCQ and QUC. So that the triangles CUQ and CRP will be ſimilar. And after the ſame manner you may demonſtrate that the triangles SLQ, STF are ſimilar; whence,

CR : PR :: QU : CU,
and ST : PT :: QL : SL.

Suppoſe CS $= a$, CA $= b$, the ſine of the angle FCO to its coſine as d to a; ſin. angle CAE to coſin. as c to a, and ſin. KSH to coſin. as e to a. Put alſo PM $= y$, CM $= x$, QN $= z$.

Then RM : PM :: $a : d$. PR : PM :: $\sqrt{a^3+d^2} : d$. AN : QN :: $a : c$. So that RM $= \frac{ay}{d}$, CR ($=$ CM $-$ RM) $= x - \frac{ay}{d}$, PR $= y\frac{\sqrt{a^2+d^2}}{d}$.

Likewiſe QU $= \frac{z\sqrt{a^2+d^2}}{d}$, and CU ($=$ CA $-$ AN $-$ NU) $= b - \frac{a}{c}z - \frac{a}{d}z$. And it being CR : PR :: QU : CU, it follows that $dx - ay : y\sqrt{a^2+d^2} :: z\sqrt{a^2+d^2} : b - \overline{\frac{a}{c} + \frac{a}{d}}z$. So that $z = \frac{bc\times\overline{dx-ay}}{\overline{dc-a^2}\times y + \overline{d+c}\times ax}$.

In like manner you will find ST $= a - x - \frac{a}{e}y$, PT $= \frac{y\sqrt{a^2+e^2}}{e}$, QL $= \frac{z\sqrt{a^2+e^2}}{e}$, and SL ($=$ AN $-$ AS $-$ NL) $= a - b + \frac{\overline{e-c}\times az}{ec}$. But it was ST : PT :: QL : SL, that is, $a-x-\frac{a}{e}y : \frac{y\sqrt{a^2+e^2}}{e} :: \frac{z\sqrt{a^2+e^2}}{e} : a-b+\frac{\overline{e-c}\times az}{ec}$.

Whence QN $= z = \dfrac{\overline{a-b}\times c\times\overline{ae-ex-ay}}{\overline{ec+a^2}\times y+ax\times\overline{e-c}+a^2\times\overline{c-e}}$.

And from the equation of theſe two values of z this equation reſults;

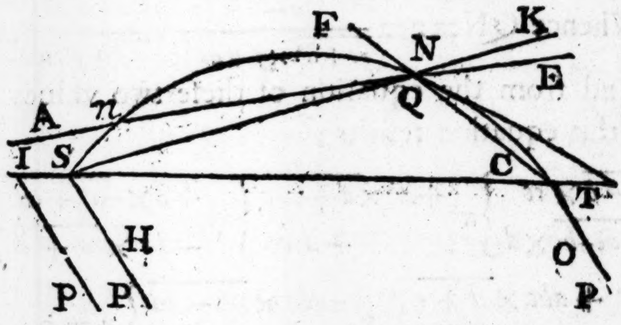
$$\left.\begin{array}{l}\overline{a-b}\times ce\\ +\overline{ae-bc}\times d\end{array}\right\}x^2 \left.\begin{array}{r}+a^2\times\overline{d+c-e}\\ +dce\end{array}\right\}xy \left.\begin{array}{r}+a\times\overline{-a^2+cd}\\ -bc\times\overline{e+d}\end{array}\right\}y^2$$

$$\left.\begin{array}{l}+abc\times\overline{d+e}\\ -a^2e\times\overline{d+c}\end{array}\right\}x \left.\begin{array}{l}-bc\times\overline{a^2-ed}\\ -ae\times\overline{dc-a^2}\end{array}\right\}y=0;$$

where ſince x and y are only of two dimenſions, it appears that the curve deſcribed muſt be a line of the ſecond *order*, or a conic ſection, according to what has been already demonſtrated.

§ 37. As the angles FCO, KSH revolve about the poles C and S, if the angle CQS becomes equal to the ſupplement of theſe given angles to four right ones, then the angle CPS muſt vaniſh, that is, the lines CO and SH muſt become parallel: and the interſection P muſt go off to an infinite diſtance. And the lines CO and SH become, in that caſe, parallel to one of the *aſymptotes*.

In order to determine if this may be, deſcribe on CS an arc of a circle that can have inſcribed in it an angle equal to the ſupplement of the angles FCO, KSH, to four right angles. If this arc meet the line AE in two points N, *n*, then when Q the interſection of the ſides CF, SK comes to either of theſe points, as it is carried along the line AE, the point P will go off to infinity, and the lines SH, CO, become parallel to each other and to an *aſymptote* of the curve.



If that arc only touch the line AE, the point P will go off to infinity but once. If the arc neither cut the line AE nor touch it, the point P cannot go off to infinity. In the firſt caſe the conic ſection is an *hyperbola*, in the ſecond a *parabola*, in the third an *ellipſe*.

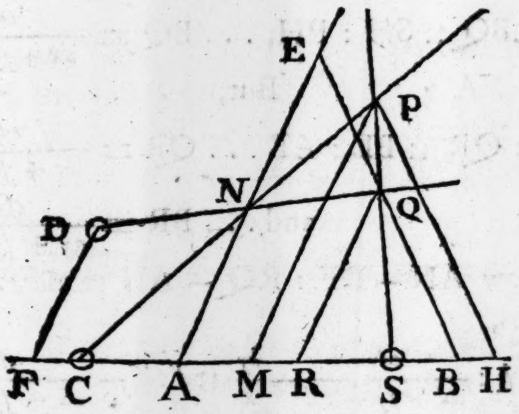
The *aſymptotes*, when the curve has any, are determined by the following conſtruction.

Draw NT conſtituitng the angle CNT = SNA, meeting SC in T; then take SI = CT, and always towards oppoſite parts, and through I draw IP parallel to SH or CO, and IP will be one aſymptote of the curve. The other is determined in like manner, by bringing Q to *n*.

And the two aſymptotes meet in the center, conſtituting there an angle $=$ NS*n*.

From this conſtruction it is obvious, that when the circular arc CN*n*S touches the line AE, the angle SNA being then $=$ SCN, the line NT will become parallel to CS; and therefore CT and SI become infinite, that is, the *aſymptote* IP going off to infinity, the curve becomes a *parabola*.

§ 38. There is another general method of deſcribing the lines of the *ſecond order*, that deſerves our conſideration.



Inſtead of angles we now uſe three rulers DQ, CN, SP, which we ſuppoſe to revolve about the poles D, C, S, and cut one another always in three points N, Q and P; and carrying any two of theſe interſections, as N and Q, along the given ſtraight lines AE, BE, the third interſection P will deſcribe a conic ſection.

Through

Through the points D, P, Q, draw DF, PM, QR, parallel to AE, meeting CS in F, M, R; alſo through P draw PH parallel to BE meeting CS in H.

Then putting $PM = y$, $CM = x$, $CS = a$, $CA = b$, $SB = c$, $DF = k$, $AF = l$, $AE = d$, $BE = e$, $AB\ (= a - b + c) = f$; ſince the triangles PMH, AEB are ſimilar, therefore $PH = \frac{ey}{d}$, $MH = \frac{fy}{d}$, $SH = \frac{dx + fy - ad}{d}$. And ſince $CA : AN :: CM : PM, \ldots AN = \frac{by}{x}$; and ſince

$$SB : BQ :: SH : PH, \ldots BQ = \frac{cey}{dx + fy - ad},$$

But,

$$BQ : QR :: BE : AE \ldots QR = \frac{cdy}{dx + fy - ad},$$

$$\text{and} \ldots BR = \frac{cfy}{dx + fy - ad}.$$

Now $AN - DF : RQ - AN :: AF : AR$; this is,

$$\frac{by}{x} - k : \frac{cdy}{dx + fy - ad} - \frac{by}{x} :: l : f - \frac{cfy}{dx + fy - ad}.$$

And multiplying the extremes and means, and ordering the terms, it is,

$$\left.\begin{array}{l} bf \times \overline{c - l - f} \times y^2 + c \times \overline{ld - kf - bd \times \overline{l + f} + kff} \times xy \\ + bad \times \overline{l + f} \times y - adfk \times x + dfk \times x^2 \end{array}\right\} = 0.$$

In which equation, the ſign of ſome terms ay vary by varying the ſituation of the poles and lines; but x and y not riſing to more than two dimenſions, it appears that the point P always deſcribes a *conic ſection.* Only in ſome particular caſes the conic ſection becomes a ſtraight line. As for example, when D is found in the ſtraight line CS; for then DF vaniſhing the terms $dfkx^2 - adfkx$ vaniſh, and the remaining terms being diviſible by y, the equation becomes,

$$bf\times\overline{c-l-f}\times y+\overline{cld-bd\times\overline{l+f}}\times x+bad\times\overline{l+f}=0,$$

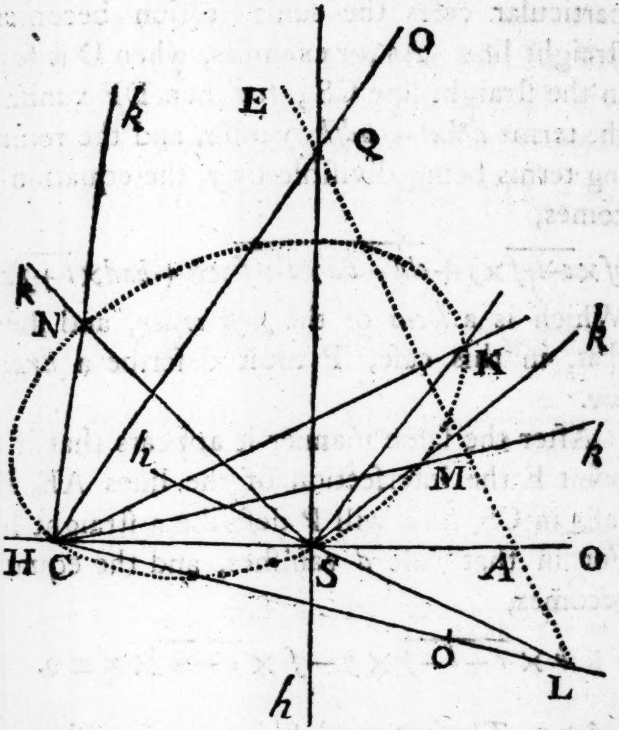
Which is a *locus* of the *firſt order*, and ſhews, that, in this caſe, P muſt deſcribe a *ſtraight line.*

After the ſame manner it appears that if the point E the interſection of the lines AE, BE, falls in CS, then will P deſcribe a ſtraight line. For in that caſe d vaniſhes, and the equation becomes,

$$b\times\overline{c-l-f}\times y-f\times\overline{c-k}\times x=0.$$

§ 39. Theſe two deſcriptions furniſh, each, a general method of "deſcribing a line of the ſecond order through any five given points whereof three are not in the ſame ſtraight line."

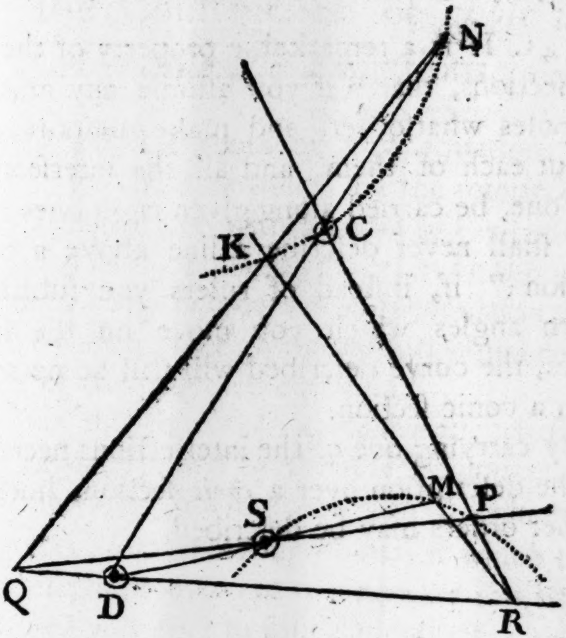
Suppoſe the five given points are C, S, M, K, N; join any three of them, as C, S, K, and let angles revolve about C and S equal to the angles KCS, KSC. Apply the interſection of the legs CK, SK firſt to the point N, and let the interſection of the legs CO and SH be Q; ſecondly apply the interſection of the ſame legs CK, SK, to the remaining point M, and let the



interſection of the legs CO, SH be L. Draw a line joining Q and L, and it will be the line AE along which if you carry the interſection of the legs CO, SH, the interſection of the other legs will deſcribe a conic ſection paſſing through the five given points C, S, M, K, N.

It muſt paſs through C and S from the conſtruction: when the interſection of CO, SH comes to A, the curve will paſs through K.

And when it becomes to Q and L, it paſſes through N, M.



§ 40. From the ſecond deſcription we have this ſolution of the ſame problem.

Let C, S, M, K, N be the five given points; draw lines joining them; produce two of the lines NC, MS, till they meet in D. Let three rulers revolve about the three poles C, S, D, *viz.* CP, SQ, DR. Let the interſection of the rulers CP, DR, be carried over the given line MK, and the interſection of the rulers SQ, DR be carried through the line NK; and the point P, the interſection of the rulers that revolve about C and S, will deſcribe a conic ſection that paſſes through the five points C, S, M, K, N.

§ 41. It is a remarkable property of the conic ſections, that "if you aſſume any number of poles whatſoever, and make rulers revolve about each of them, and all the interſections but one, be carried along given right lines, that one ſhall never deſcribe a line above a conic ſection;" if, inſtead of rulers you ſubſtitute given angles which you move on the ſame poles, the curve deſcribed will ſtill be no more then a conic ſection.

By carrying one of the interſections neceſſary in the deſcription over a *conic* ſection, lines of higher orders may be deſcribed.

◆

CHAP.

CHAP. III.

Of the Conſtruction of *cubic* and *biquadratic* Equations.

42. "THE roots of any equation may be determined by the interſections of a ſtraight line with a curve of the ſame dimenſions as the equation:" or, "by the interſections of any two curves whoſe indices multiplied by each other give a product equal to the index of the propoſed equation."

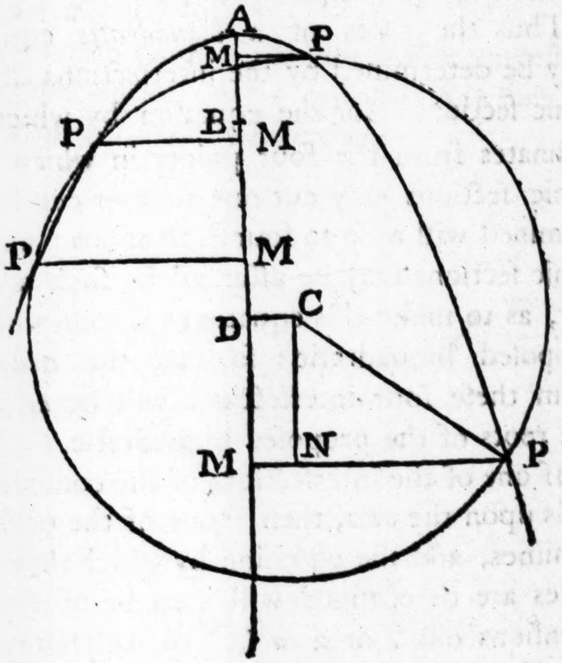
Thus the roots of a *biquadratic* equation may be determined by the interſections of two conic ſections; for the equation by which the ordinates from the four points in which theſe conic ſections may cut one another can be determined will ariſe to four dimenſions: and the conic ſections may be aſſumed in ſuch a manner, as to make this equation coincide with any propoſed biquadratic: ſo that the ordinates from theſe four interſections will be equal to the roots of the propoſed biquadratic.

If one of the interſections of the conic ſection falls upon the *axis*, then "one of the ordinates vaniſhes, and the equation by which theſe ordinates are determined will then be of three dimenſions only, or a *cubic*," to which any propoſed cubic equation may be accommodated.

So that the three remaining ordinates will be the three roots of that propoſed cubic.

§ 43. Thoſe conic ſections ought to be preferred for this purpoſe that are moſt eaſily deſcribed. They muſt not however be both *circles*; for their interſections are only two, and can ſerve only for the reſolution of *quadratic* equations.

Yet the circle ought to be one, as being moſt eaſily deſcribed; and the *parabola* is commonly aſſumed for the other. Their interſections are determined in the following manner.



Let APE be the common Apollonian parabola. Take on its axis the line AB = half of its parameter. Let C be any point in the plane of the parabola, and from it as a center deſcribe, with any radius CP, a circle meeting the parabola in P. Let PM, CD, be perpendiculars on the axis in M and D, and let CN, parallel to the axis, meet PM in N.

Then will always CPq = CNq + NPq (47 é 1.)

Put CP $= a$, the parameter of the parabola $= b$, AD $= c$, DC $= d$, AM $= x$, PM $= y$.

Then CN$q = \overline{x \overset{\backsim}{+} c}|^2$, NP$q = \overline{y \overset{\backsim}{+} d}|^2$; and $\overline{x \overset{\backsim}{+} c}|^2 + \overline{y \overset{\backsim}{+} d}|^2 = a^2$*. That is,

$$x^2 \pm 2cx + c^2 + y^2 \pm 2dy + d^2 = a^2.$$

But, from the nature of the parabola, $y^2 = bx$, and $x^2 = \frac{y^4}{b^2}$; ſubſtituting therefore theſe values for x^2 and x, it will be,

$$\frac{y^4}{b^2} \pm \frac{2cy^2}{b} + y^2 \pm 2dy + c^2 + d^2 - a^2 = 0.$$

Or, multiplying by b^2,

$$y^4 \pm \overline{2bc + b^2} \times y^2 \pm 2db^2 \times y + \overline{c^2 + d^2 - a^2} \times b^2 = 0.$$

Which may repreſent any biquadratic equation that wants the ſecond term; ſince ſuch values may be found for a, b, c, and d, by comparing this with any propoſed biquadratic, as to make them coincide. And then the ordinates from the points P, P, P, P, on the axis will be equal to the roots of that propoſed biquadratic. And this may be done, though the parameter of the parabola (*viz.* b) be given: that is, if you have a parabola already made or given, by it alone you may reſolve all biquadratic equations, and you will only need to vary the center of your circle and its radius.

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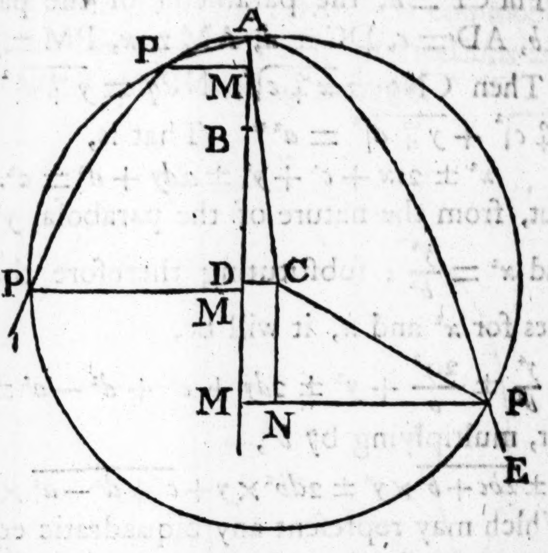
* $x \backsim c$ is the differences of x and c indefinitely, whichever of the two is greateſt.

§ 44. If the circle deſcribed from the center C paſs through the vertex A, then $CPq = CAq = CDq + ADq$, that is, $a^2 = d^2 + c^2$; and the last term of the biquadratic $(c^2 + d^2 - a^2)$ will vaniſh; therefore, dividing the reſt by y, there ariſes the cubic,



$$y^{3*} \pm \overline{2bc + b^2} \times y \pm 2db^2 = 0.$$

Let the cubic equation propoſed to be reſolved be $y^{3*} \pm py \pm r = 0$. Compare the terms of theſe two equations, and you will have $\pm 2bc + b^2 = \pm p$, and $\pm 2db^2 = \pm r$, or, $\mp c = \frac{b}{2} \mp \frac{p}{2b}$, and $d = \pm \frac{r}{2b^2}$. From which you have this conſtruction of the cubic $y^{3*} \pm py \pm r = 0$, by means of any given parabola APE.

> "*From the point B take in the axis (forward if the equation has* $-p$, *but backwards if* p *is poſitive) the line* $BD = \frac{p}{2b}$; *then raiſe the perpendicular* $DC = \frac{r}{2b^2}$, *and from C, deſcribe a circle paſſing through the vertex A, meeting the paralola in P, ſo ſhall the ordinate PM be one of the roots of the cubic* $y^{3*} \pm py \pm r = 0$."

The ordinates that ſtand on the ſame ſide of the axis with the center C are negative or affirmative, according to the laſt term r is *negative* or *affirmative*; and thoſe ordinates have always *contrary* ſigns that ſtand on *different* ſides of the *axis*. The roots are found of the ſame value, only they have contrary ſigns, when r is poſitive as when it is negative; the ſecond term of the equation being wanting; which agrees with what has been demonſtrated elſewhere.

§ 45. In reſolving *numerical* equations, you may ſuppoſe the parameter b to be unit; then

$AD = \frac{1}{2} \mp \frac{1}{2}p$, and $DC = \frac{1}{2}r$; and the ordinate PM muſt then be meaſured on a ſcale where the *parameter*, or 2AB is unit. Or, if it be more convenient, the *parameter* may be ſuppoſed to expreſs 10, 100, *&c.* or any other number, and PM will be found by meaſuring it on a ſcale where the parameter is 10, 100, *&c.* or that other number.

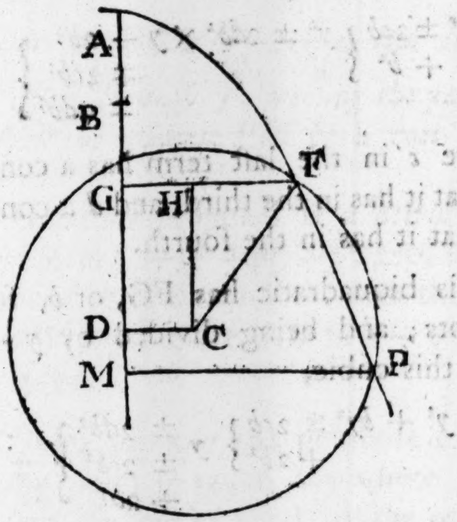
§ 46. "When the circle meets the *parabola* in one point only beſides the vertex, the equation has only one real root, and the other two imaginary."

Thus, if the equation has $+p$, or if D falls on the ſame ſide of B as A does, the circle can meet the parabola in two points only, whereof A is one; and therefore the equation muſt have two *imaginary* roots; as we demonſtrated elſewhere. If the circle *touch* the *parabola*, then two roots of the equation are equal.

It is alſo obvious, that the equation muſt neceſſarily have *one real* root; becauſe, ſince the circle meets the *parabola* in the vertex A, it muſt meet it in one other point, at leaſt, beſides A.

§ 47. Inſtead of making the circle paſs thro' the *vertex* A, you may ſuppoſe it to paſs through ſome other given point in the parabola, and that inſerſection being given, the *biquadratic* found for determining the interſections, in § 43, may be reduced to a *cubic*.

Let the ordinate belonging to that given



interſection be g, then one of the values of y being g, it follows that the biquadratic

$$\left.\begin{array}{l} y^{4*} \pm 2bc \\ \quad + b^2 \end{array}\right\} y^2 \pm 2db^2 y + \overline{d^2 + c^2 - a^2} \times b^2 = 0$$

will be diviſible by $y - g$, which will reduce it to a *cubic* that ſhall have the ſecond term. And thus we have a conſtruction for *cubic* equations that have all their terms.

For example, let us ſuppoſe that the *parameter* is AG, and the ordinate at G is GF meeting the curve in F. Suppoſe now that the circle is always to paſs through F; then ſhall $\mathrm{CF}q\,(= a^2) = \mathrm{CH}q + \mathrm{HF}q = \overline{c \underset{\backsim}{+} b}|^2 + \overline{b \underset{\backsim}{+} d}|^2$ $= c^2 + d^2 \pm 2cb \pm 2db + 2b^2$, and ſubſtituting in the equation of § 43 this value of a^2, it becomes

$$\left.\begin{matrix} y^{4*} \pm 2cb \\ +b^2 \end{matrix}\right\} y^2 \pm 2db^2 \times y \left.\begin{matrix} -2b^4 \\ \mp 2cb^3 \\ \mp 2db^3 \end{matrix}\right\} = 0.$$

Where c in the laſt term has a contrary ſign to what it has in the third, and d a contrary ſign to what it has in the fourth.

This biquadratic has FG, or b, for one of its roots; and being divided by $y - b$, there ariſes this cubic,

$$y^3 + by^2 \left.\begin{matrix} \pm 2cb \\ +2b^2 \end{matrix}\right\} y \left.\begin{matrix} \pm 2db^2 \\ \pm 2cb^2 \\ +2b^3 \end{matrix}\right\} = 0,$$

having all its terms compleat. If C had been taken on the other ſide of the axis, the ſecond term by^2 had been negative.

Let now any cubic equation be propoſed to be reſolved, as $y^3 + py^2 + qy - r = 0$. And, by comparing it with the preceding, you will find

$$\left.\begin{matrix} p = b \\ q = 2b^2 \pm 2bc \\ -r = 2b^3 \pm 2cb^2 \pm 2db^2 \end{matrix}\right\} \text{whence} \left\{\begin{matrix} b = p, \\ \pm c = p - \frac{q}{2p}, \\ d = \frac{q}{2p} + \frac{r}{2p^2}. \end{matrix}\right.$$

Therefore, to conſtruct the propoſed cubic *equation* $y^3 + py^2 + qy - r = 0$, *let the parameter of your* parabola *be equal to* p, *take, on the axis from the*

the vertex A, the line $AD = p - \frac{q}{2p}$, *and raiſe the perpendicular* $DC = \frac{q}{2p} + \frac{r}{2p^2}$, *and from C deſcribe a circle through F, meeting the parabola in P, ſo ſhall the ordinate PM be a root of the equation.*

If the equation propoſed is a *literal* equation of this form $y^3+py^2+pqy-p^2r=0$, having all the terms of three dimenſions, then this conſtruction will only require $AD = 1 - \frac{1}{2}q$, and $DC = \frac{1}{2}q + \frac{1}{2}r$.

§ 48. If you ſuppoſe the parabola to paſs through any point F taken any where in the parabola *(vid. Fig. preced.)* and call the ordinate $FG = e$, then $\overline{c - \frac{e^2}{b}}\big|^2 + \overline{e-d}\big|^2 = a^2$, and the general biquadratic may have this form,

$$y^{4*} \left.\begin{array}{l} +2cb \\ +b^2 \end{array}\right\} y^2 - 2db^2y \left.\begin{array}{l} +2ce^2b \\ +2deb^2 \\ -e^4 \\ -e^2b^2 \end{array}\right\} = 0.$$

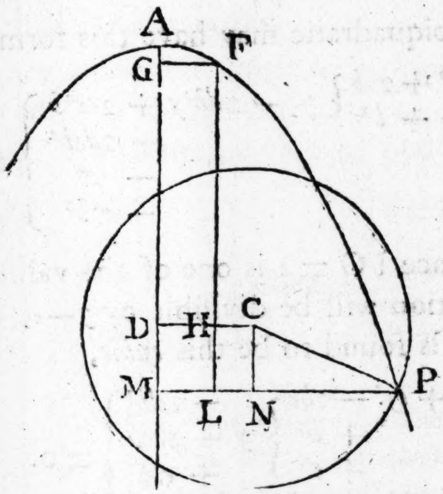
But ſince $FG = e$ is one of the values of y, the equation will be diviſible by $y-e$, and the quotient is found to be this *cubic*,

$$y^3 + ey^2 \left.\begin{array}{l} -2bc \\ +b^2 \\ +e^2 \end{array}\right\} y \left.\begin{array}{l} -2db^2 \\ -2ceb \\ +eb^2 \\ +e^3 \end{array}\right\} = 0.$$

Which compared with $y^3 + py^2 + pqy - p^2r = 0$, gives FG (or e) $= p$, AD ($= c$) $= \frac{p^2 + b^2 - pq}{2b}$, and DC $= d = \frac{p^2 \times \overline{q + r}}{2b^2}$. And by this conſtruction the roots of a compleat cubic equation may be found by any parabola whatſoever.

§ 42. It is eaſy to ſee from § 43. how to conſtruct the roots of a biquadratic by any parabola, after the ſecond term is taken away. But "the roots of a biquadratic may be determined by any parabola;" only they cannot be the ordinates on the axis, but "may be equal to the perpendiculars on a line parallel to the axis, meeting the parabola in F, CD in H, and PM in L."

Let FG be an ordinate to the axis in G; and



the reſt remaining as before, let FL $= x$, PL $= y$, the parameter $= b$, CP $= a$, FH $= c$, CH $= d$, FG $= e$.

And ſince PMq $(= \overline{PL + HD}|^{q}) = AM \times b$, therefore $y^2 + 2ey + e^2 = \overline{AG + FL} \times b = \overline{\frac{e^2}{b} + x} \times b$; and conſequently $y^2 + 2ey = bx$.

But CNq + NPq = CPq; that is, $\overline{x - c}|^2 + \overline{y - d}|^2 = a^2$. And ſubſtituting for x^2 and x their values $\frac{\overline{y^2 + 2ey}|^2}{b^2}$ and $\frac{y^2 + 2ey}{b}$; you will find

$$y^4 + 4ey^3 \left.\begin{array}{l} + 4e^2 \\ - 2cb \\ + b^2 \end{array}\right\} y^2 \left.\begin{array}{l} - 4ecb \\ - 2db^2 \end{array}\right\} y \left.\begin{array}{l} + c^2b^2 \\ + d^2b^2 \\ + a^2b^2 \end{array}\right\} = 0,$$

which is a complete biquadratic equation. And by comparing with it the equation $y^4 + py^3 + bqy^2 - b^2ry - b^3s = 0$, you will find FG $(= e) = \frac{1}{4}p$, FH $(= c) = \frac{\frac{1}{4}p^2 + b^2 - bq}{2b}$, HC $(= d) = \frac{b^2r - pbc}{2b^2}$, and CP $(= a) = \sqrt{b^3s + c^2 + d^2}$: which gives a general conſtruction for any ſuch biquadratic equation by any parabola whatſoever. If the ſigns of p, q, r, or s, are different, it is eaſy to make the neceſſary alterations in the conſtruction. *Ex. gr.* If p is negative, then FG muſt be taken on the other ſide of the axis.

If you ſuppoſe the circle to paſs through F, the equation will become a *cubic* having all its terms: the laſt term $\overline{c^2 + d^2 - a^2} \times b^2$ vaniſhing, becauſe then $c^2 + d^2 = a^2$. It will have this form,

$$y^3 + 4ey^2 + \left.\begin{matrix} 4e^2 \\ -2cb \\ +b^2 \end{matrix}\right\} y \left.\begin{matrix} -4ecb \\ -2db^2 \end{matrix}\right\} = 0;$$

and then "the conſtruction will give the roots of a compleat cubic equation."

§ 50. We have ſufficiently ſhewed, how the roots of cubic and biquadratic equations may be conſtructed by the parabola and circle; we ſhall now ſhew how *other* conic ſections may be determined by whoſe interſections the ſame roots may be diſcovered.

Let the equation propoſed be $y^4 + bpy^2 + b^2qy - b^3r = 0$; and let us ſuppoſe, that,

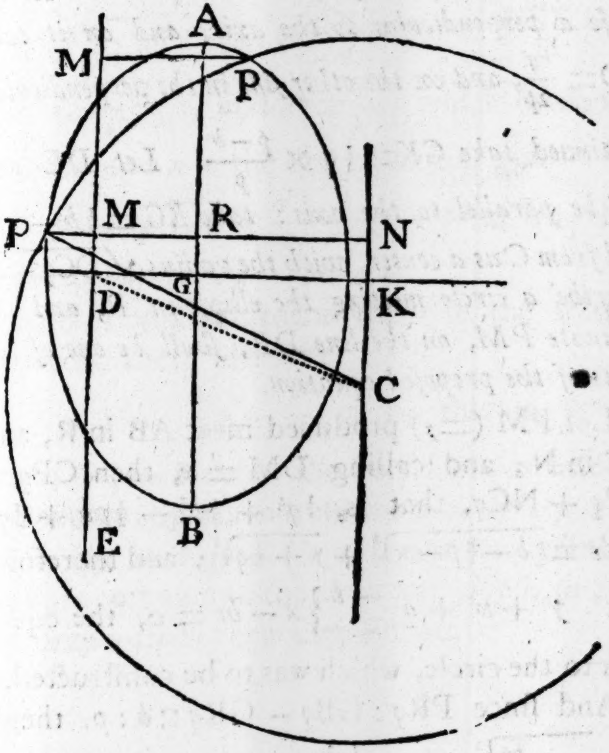
1°. $bx = y^2$; then ſhall we have by ſubſtitution of b^2x^2 for y^4, and dividing by bp,

2°. $y^2 + \frac{b}{p}x^2 + \frac{bq}{p}y - \frac{b^2r}{p} = 0$, which has its *locus* an *ellipſe*. Then by ſubſtituting (in this laſt) bx for y^2, and multiplying all the terms by $\frac{p}{b}$, you find,

3°. $x^2 + px + qy - br = 0$, an equation to a *parabola*. Then, adding to this equation $y^2 - bx = 0$, you will have,

4°. $x^2 + y^2 \left.\begin{matrix} +p \\ -b \end{matrix}\right\} x + qy - br = 0$, an equation to a *circle*.

The roots of the equation $y^{4*} + bpy^2 + b^2qy - b^3r = 0$ may be determined by the interſection of any *two* of theſe *loci*; as for example,



by the interſections of the *ellipſe* that is the *locus* of the equation $y^2 + \frac{b}{p}x^2 + \frac{bq}{p}y - \frac{b^2r}{p} = 0$, and of the *circle* which is the *locus* of $x^2 + y^2 \left.\begin{matrix}+p\\-b\end{matrix}\right\} x + qy - br = 0$, from which we deduce this conſtruction.

Let AB be the axis of an ellipſis, equal to $\sqrt{br+\frac{bq^2}{4p}}$, *let G be the center of the ellipſe, and the axis to the parameter as p to b. At G, raiſe a perpendicular to the axis, and on it take* $GD=\frac{bq}{2p}$, *and on the other ſide in the perpendicular continued take* $GK=\frac{1}{2}q\times\frac{p-b}{p}$. *Let DE and KC be parallel to the axis: take* $KC=\frac{1}{2}b-\frac{1}{2}p$, *and from C as a center, with the radius* $\sqrt{DCq+br}$ *deſcribe a circle meeting the ellipſe in P, and the ordinate PM, on the line DE, ſhall be one of the roots of the propoſed equation.*

Let PM ($=y$) produced meet AB in R, and KC in N; and calling $DM=x$, then $CPq=NPq+NCq$, that is, $\frac{1}{4}q^2+\frac{1}{4}b^2-\frac{1}{2}pb+\frac{1}{4}p^2+br=\overline{\tfrac{1}{2}b-\tfrac{1}{2}p-x}\big|^2+\overline{y+\tfrac{1}{2}q}\big|^2$; and therefore,

1°. $y^2+x^2+qy\begin{matrix}-b\\+p\end{matrix}\Big\}x-br=0$, the equation to the circle, which was to be conſtructed.

And ſince $PRq:GBq-GRq::b:p$, therefore $\overline{y+\frac{bq}{2p}}\Big|^2:br+\frac{bq^2}{4p}-x^2::b:p$; and conſequently,

2°. $y^2+\frac{b}{p}x^2+\frac{bq}{p}y-\frac{b^2r}{p}=0$; which is the equation that was to be conſtructed.

Now that their interſections will give the roots required, appears thus.

For x^2 in the firſt equation ſubſtitute the value you deduce for it from the ſecond, *viz.* $br - \frac{p}{b}y^2 - qy$, and there will ariſe

$y^2 - \frac{p}{b}y^2 \left.\begin{matrix}-b\\+p\end{matrix}\right\} x = 0$, or $\frac{b-p}{b} \times y^2 = \overline{b-p} \times x$, that is, $\frac{y^2}{b} = x$, and $x^2 = \frac{y^4}{b^2}$; which ſubſtituted for x^2 and x in the firſt equation, gives $\frac{y^4}{b^2} + y^2 + \overline{p-b} \times \frac{y^2}{b} + qy - br = 0$; that is, $y^{4*} + bpy^2 + b^2qy - b^3r = 0$.

And if you ſubſtitute them in the ſecond equation, there will ariſe $\frac{b}{pb^3}y^4 + y^2 + \frac{bq}{p}y - \frac{b^2r}{p} = 0$, that is, $y^{4*} + bpy^2 + b^2qy - b^3r = 0$, the very ſame as before; and thus it appears that the roots of the equation $y^{4*} - bpy^2 + b^2qy - b^3r = 0$ are the ordinates that are common to the *circle* and *ellipſe*, or that are drawn from their interſection.

End of the THIRD PART.

APPENDIX:

DE

Linearum Geometricarum

Proprietatibus generalibus

TRACTATUS.

DE
LINEARUM GEOMETRICARUM
Proprietatibus generalibus.

DE lineis ſecundi ordinis, ſive ſectionibus conicis, ſcripſerunt uberrime geometræ veteres & recentiores; de figuris quæ ad ſuperiores linearum ordines referuntur pauca & exilia tantum ante NEWTONUM tradiderunt. Vir illuſtriſſimus, in Tractatu de *Enumeratione Linearum tertii Ordinis*, doctrinam hanc, cum diu jacuiſſet, excitavit, dignamque eſſe in qua elaborarent geometris oſtendit. Expoſitis enim harum linearum proprietatibus generalibus, quæ vulgatis ſectionum conicarum affectionibus ſunt adeo affines ut velut ad eandem normam compoſitæ videantur, alios ſuo exemplo impulit ut analogiam hanc ſive ſimilitudinem quæ tam diverſis intercedit figurarum generibus bene cognitam & ſatis firme animo conceptam atque comprehenſam habere ſtuderent. In qua illuſtranda & ulterius indaganda curam operamque merito poſuerunt; cum nihil ſit omnium quæ in diſciplinis purè mathematicis tractantur quod pulchrius dicatur aut ad animum veri inveſtigandi cupidum oblectandum aptius, quam rerum tam diverſarum conſenſus ſive harmonia, ipſiuſque doctrinæ compoſitio & nexus admirabilis, quo poſterius priori convenit, quod ſequitur ſuperiori reſpondet, quæque ſimpliciora ſunt ad magis ardua viam conſtanter aperiunt.

Linearum tertii ordinis proprietates generales a *Newtono* traditæ parallelarum ſegmenta & aſymptotos pleræque ſpectant. Alias harum affectiones quaſdam diverſi generis breviter indicavimus in tractatu de fluxionibus nuper edito, Art. 324, & 401. Celeberrimus *Coteſius* pulcherrimam olim detexit linearum geometricarum proprietatem, hucuſque ineditam, quam abſque demonſtratione nobis communicavit vir Reverendus D. *Robertus Smith*, Collegii S. S. Trinitatis apud Cantabrigienſes Præfectus, doctrina operibuſque ſuis pariter ac fide & ſtudio in amicos clarus. De his meditantibus nobis alia quoque ſe obtulerunt theoremata generalia; quæ cum ad arduam hanc geometriæ partem augendam & illuſtrandam conducere viderentur, ipſa quaſi in faſciculum congerenda & una ſerie breviter exponenda & demonſtranda putavimus.

* * *

SECTIO I.

De Lineis Geometricis in genere.

§ 1. LIneæ ſecundi ordinis ſectione ſolidi geometrici, coni ſcilicet, definiuntur, unde earum proprietates per vulgarem geometriam optime derivantur. Verum diverſa eſt ratio figurarum quæ ad ſuperiores linearum ordines referuntur. Ad has definiendas, earumque proprietates eruendas, adhibendæ ſunt æquationes generales co-ordinatarum relationem exprimentes. Repræſentet x abſciſſam AP, y ordinatam PM figuræ FMH, Fig. 1. denotentque a, b, c, d, e, &c. coefficientes quaſcunque invariabiles; & dato angulo APM ſi relatio co-ordinatarum x & y definiatur æquatione quæ, præter ipſas co-ordinatas, ſolas involvat coefficientes invariabiles, linea FMH geometrica appellatur; quæ quidem auctoribus quibuſdam linea algebraica, aliis linea rationalis dicitur. Ordo autem lineæ pendet ab indice altiſſimo ipſius x vel y in terminis æquationis a fractionibus & ſurdis liberatæ, vel a ſumma indicis utriuſque in termino ubi hæc ſumma prodit maxima. Termini enim x^2, xy, y^2 ad ſecundum ordinem pariter referuntur; termini x^3, x^2y, xy^2, y^3 ad tertium. Itaque æquatio $y=ax+b$, ſive $y-ax-b=0$, eſt primi ordinis & deſignat lineam ſive locum primi ordinis, quæ quidem ſemper recta eſt. Sumatur enim in ordinata PM recta PN ita ut PN ſit ad AP ut $+a$ ad unitatem; conſtituatur AD parallela ordinatæ PM æqualis ipſi $+b$, & ducta DM parallela rectæ AN erit locus cui æquatio propoſita reſpondebit. Nam $PM = PN + NM = (a \times AP + AD)\ ax + b$. Quod ſi æquatio ſit formæ $y = ax - b$ vel $y = -ax + b$, recta AD, vel PN, ſumenda eſt ad alteram partem abſciſſæ AP; contrarius enim rectarum ſitus contrariis coefficientium ſignis reſpondet. Si valores affirmativi ipſius x deſignent rectas ad dextram ductas a principio abſciſſæ A, valores negativi denotabunt rectas ab eodem principio ad ſiniſtram ductas; & ſimiliter ſi valores affirmativi ipſius y ordinatas repræſentent ſupra abſciſſam conſtitutas, negativi deſignabunt ordinatas infra abſciſſam ad oppoſitas partes ductas. Fig. 2.

Æquatio generalis ad lineam ſecundi ordinis eſt hujus formæ

$$\begin{array}{l} yy - axy + cx^2 = 0 \\ \quad - by - dx \\ \qquad + e \end{array}$$

& æquatio generalis ad lineas tertii ordinis eſt $y^3 - \overline{ax + b} \times y^2 + \overline{cxx - dx + e} \times y - fx^3 + gx^2 - hx + k = 0$. Et ſimilibus æquationibus definiuntur lineæ geometricæ ſuperiorum ordinum.

§ 2. Linea geometrica occurrere poteſt rectæ in tot punctis quot ſunt unitates in numero qui æquationis vel lineæ ordinem deſignat, & nunquam in pluribus. Occurſus curvæ & abſciſſæ AP definiuntur ponendo $y = 0$, quo in caſu reſtat tantum ultimus æquationis terminus quem y non ingreditur. Linea tertii ordinis *ex. gr.* occurrit abſciſſæ AP cum $fx^3 - gx^2 + hx - k = 0$, cujus æquationis ſi tres radices ſint reales abſciſſa ſecabit curvam in tribus punctis. Similiter in æquatione generali cujuſcunque ordinis index altiſſimus abſciſſæ x æqualis eſt numero qui lineæ ordinem deſignat, ſed nunquam major, adeoque is eſt numerus maximus occurſuum curvæ cum abſciſſa vel alia quavis rectâ. Cum autem æquationis cubicæ unica ſaltem radix ſit ſemper realis, idemque conſtet de æquatione quavis quinti aut imparis cujuſvis ordinis (quoniam radix quævis imaginaria aliam neceſſario ſemper habet comitem), ſequitur lineam tertii aut imparis cujuſcunque ordinis rectam quamvis aſymptoto non parallelam in eodem plano ductam in uno ſaltem puncto neceſſario ſecare. Si vero recta ſit aſymptoto parallela, in hoc caſu vulgo dicitur curvæ occurrere ad diſtantiam infinitam. Linea igitur imparis cujuſcunque ordinis duo ſaltem habet crura in infinitum progredientia. Æquationis autem quadraticæ vel paris cujuſvis ordinis radices omnes nonnunquam fiunt imaginariæ, adeoque fieri poteſt ut recta in plano lineæ paris ordinis ducta eidem nullibi occurrat.

§ 3. Æquatio ſecundi aut ſuperioris cujuſcunque ordinis quandoque componitur ex tot ſimplicibus, a ſurdis & fractis liberatis, in ſe mutuo ductis quot ſunt ipſius æquationis propoſitæ dimenſiones; quo in caſu figura FMH non eſt curvilinea ſed conflatur ex totidem rectis, quæ per ſimplices has æquationes definiuntur ut in *Art.* 1. Similiter ſi æquatio cubica componatur ex æquationibus duabus in ſe mutuo ductis, quarum altera ſit quadratica altera ſimplex, locus non erit linea tertii ordinis proprie ſic dicta, ſed ſectio, conica cum rectâ adjunctâ. Proprietatis autem quæ de lineis geometricis ſuperiorum ordinum generaliter demonſtrantur, affirmandæ ſunt quoque de lineis inferiorum ordinum, modo numeri harum ordines deſignantes ſimul ſumpti numerum compleant qui ordinem dictæ ſuperioris lineæ denotat. Quæ de lineis tertii ordinis (*ex. gr.*) generaliter demonſtrantur affirmanda quoque ſunt de tribus rectis in eodem plano ductis, vel de ſectione conica cum unica quavis recta ſimul in eodem plano deſcriptis. Ex altera parte, vix ulla aſſignari poteſt proprietas lineæ ordinis inferioris ſatis generalis cui non reſpondeat affectio aliqua linearum ordinum ſuperiorum. Has autem ex illis derivare non eſt cujuſvis diligentiæ. Pendet hæc doctrina magna eſt parte a proprietatibus æquationum generalium, quas hic memorare tantum convenit.

§ 4. In æquatione quacunque coefficiens ſecundi termini æqualis eſt exceſſui quo ſumma radicum affirmativarum ſuperat ſummam negativarum; & ſi deſit hic terminus, indicio eſt ſummas radicum affirmativarum & negativarum, vel ſummas ordinatarum ad diverſas partes abſciſſæ conſtitutarum, æquales eſſe. Sit æquatio generalis ad lineam ordinis n,

$y^n - \overline{ax+b} \times y^{n-1} + \overline{cxx - dx + e} \times y^{n-2} - \&c. = 0$, ſupponatur $u = y - \frac{ax+b}{n}$, pro y ſubſtituatur ipſius valor $u + \frac{ax+b}{n}$; & in æquatione transformata deerit ſecundus terminus u^{n-1}; ut ex calculo, vel ex doctrinâ æquationum paſſim traditâ facile patet: & hinc quoque conſtat, quod per hypotheſim valor quiſque ipſius u minor ſit valore correſpondente ipſius y differentia $\frac{ax+b}{n}$; unde ſequitur ſummam valorum ipſius u (quorum numerus eſt n) deficere a ſumma valorum ipſius y (quæ ſumma eſt $ax + b$) differentia $\frac{ax+b}{n} \times n = ax + b$, adeoque priorem ſummam evaneſcere & ſecundum terminum deeſſe in æquatione qua u definitur, vel affirmativos & negativos valores ipſius u æquales ſummas conficere. Si itaque ſumatur PQ $= \frac{ax+b}{n}$, ut ſit QM $= u$, rectæ ex utraque parte puncti Q ad curvam terminatæ eandem conficient ſummam. Locus autem puncti Q eſt recta BD quæ abſciſſam ultra principium A productam ſecat in B ita ut AB $= \frac{b}{a}$, & ordinatam AD ipſi PM parallelam in D ita ut ſit AD $= \frac{1}{n} \times b$; ſi enim hæc recta ordinatæ PM occurrat in puncto Q, erit PQ ad PB (ſeu $\frac{b}{a} + x$) ut AD ad AB vel a ad n, adeoque PQ $= \frac{ax+b}{n}$, ut oportebat. Atque hinc conſtat rectam ſemper duci poſſe quæ parallelas quaſvis lineæ geometricæ occurrentes in tot punctis quot ſunt figuræ dimenſiones ita ſecabit ut ſumma ſegmentorum cujuſvis paral-

Fig. 3.

parallelæ ex una ſecantis parte ad curvam terminatorum ſemper æqualis ſit ſummæ ſegmentorum ejuſdem ex altera ſecantis parte. Manifeſtum autem eſt rectam quæ duas quaſvis parallelas hac ratione ſecat ipſam neceſſario eſſe quæ ſimiliter alias omnes parallelas ſecabit. Atque hinc patet veritas theorematis *Newtoniani*, quo continetur proprietas linearum geometricarum generalis, notiſſimæ ſectionum conicarum proprietati analoga. In his enim recta quæ duas quaſvis parallelas ad ſectionem terminatas biſecat diameter eſt, & biſecat alias omnes hiſce parallelas ad ſectionem terminatas. Et ſimiliter recta quæ duas quaſvis parallelas lineæ geometricæ occurrentes in tot punctis quot ipſa eſt dimenſionum ita ſecat ut ſumma partium ex uno ſecantis latere conſiſtentium & ad curvam terminatarum æqualis ſit ſummæ partium ejuſdem parallelæ ex altero ſecantis latere conſiſtentium ad curvam terminatarum, eodem modo ſecabit alias quaſvis rectas his parallelas.

§ 5. In æquatione quavis terminus ultimus, ſive is quem radix y non ingreditur, æqualis eſt facto ex radicibus omnibus in ſe mutuo ductis; unde ad aliam ducimur non minus generalem linearum geometricarum proprietatem. Occurrat recta PM lineæ tertii ordinis in M, m & μ, eritque $PM \times Pm \times P\mu = fx^3 - gx^2 + hx - k$. Secet abſciſſa AP curvam in tribus punctis I, K, L; & AI, AK, AL erunt valores abſciſſæ x, poſita ordinata $y = 0$, quo in caſu æquatio generalis dat $fx^3 - gx^2 + hx - k = 0$ pro his valoribus determinandis, ut in Art. 2. expoſuimus. Æquationis igitur $x^3 - \frac{gx^2}{f} + \frac{hx}{f} - \frac{k}{f} = 0$ tres radices ſunt AI, AK, AL; adeoque hæc æquatio componitur ex tribus $x - AI$, $x - AK$, $x - AL$ in ſe

Fig. 1.

ſe mutuo ductis; eſtque $x^3 - \frac{gx^2}{f} + \frac{bx}{f} - \frac{k}{f} = \overline{x - AI} \times \overline{x - AK} \times \overline{x - AL} = \overline{AP - AI} \times \overline{AP - AK} \times \overline{AP - AL} = IP \times KP \times LP = \frac{1}{f} \times PM \times Pm \times P\mu$. Factum igitur ex ordinatis PM, P*m*, P*μ* ad punctum P & curvam terminatis eſt ad factum ex ſegmentis IP, KP, LP, rectæ AP, eodem puncto & curva terminatis in ratione invariabili coefficientis *f* ad unitatem. Simili ratione demonſtratur, dato angulo APM, ſi rectæ AP, PM, lineam geometricam cujuſvis ordinis ſecent in tot punctis quot ipſa eſt dimenſionum, fore ſemper factum ex ſegmentis prioris ad punctum P & curvam terminatis ad factum ex ſegmentis poſterioris eodem puncto & curva terminatis in ratione invariabili.

§ 6. In articulo præcedente ſuppoſuimus, cum *Newtono*, rectam AP lineam tertii ordinis ſecare in tribus punctis I, K, L; verum ut theorema egregium reddatur generalius, ſupponamus abſciſſam AP in unico tantum puncto curvam ſecare; ſitque id punctum A.
Fig. 4. Quoniam igitur evaneſcente *y* evaneſcat quoque *x*, ultimus æquationis terminus, in hoc caſu, erit $fx^3 - gx^2 + bx = fx \times \overline{xx - \frac{gx}{f} + \frac{b}{f}} = fx \times \overline{\overline{x - \frac{g}{2f}}^2 + \frac{b}{f} - \frac{gg}{4ff}}$ (ſi ſumatur A*a* verſus P æqualis $\frac{g}{2f}$, & ad punctum *a* erigatur perpendicularis $ab = \frac{\sqrt{4fb - gg}}{2f}$) $= f \times AP \times \overline{aP^2 + ab^2} = f \times AP \times bP^2$; unde cum PM × P*m* × P*μ*, ſit æqualis ultimo termino $fx^3 - gx^2 + bx$, ut

ut in articulo præcedente; erit PM × Pm × Pμ ad AP × bP^2 in ratione conſtante coefficientis f ad unitatem. Valor autem rectæ perpendicularis ab eſt ſemper realis quoties recta AP curvam in unico puncto ſecat; in hoc enim caſu radices æquationis quadraticæ $fx^2 - gx + h$ ſunt neceſſario imaginariæ, adeoque $4fh$ major quam gg, & quantitas $\sqrt{4fh - gg}$ realis. Cum igitur recta quævis in unico puncto A ſecat lineam tertii ordinis, eſt ſolidum ſub ordinatis PM, Pm, Pμ ad ſolidum ſub abſciſſa AP & quadrato diſtantiæ puncti P a puncto dato b in ratione conſtanti. Juncta Ab eſt ad Aa, ſive radius ad coſinum anguli bAP, ut $\sqrt{4fh}$ ad g, & A$b = \sqrt{\frac{h}{f}}$. Idem vero punctum b ſemper convenit eidem rectæ AP, qualiſcunque ſit angulus qui abſciſſâ & ordinatâ continetur.

§ 7. Sit figura ſectio conica, cujus æquatio generalis ſit $yy - \overline{ax - b} \times y + cxx - dx + e = 0$ ut ſupra; & ſi æquationis $cxx - dx + e = 0$ radices ſint imaginariæ, recta AP ſectioni non occurret. In hoc autem caſu quantitas $4ec$ ſemper ſuperat ipſam dd; unde cum ſit $cxx - dx + e = c \times \overline{x - \frac{d}{2c}}^2 + e - \frac{dd}{4c}$ (ſi ſumatur A$a = \frac{d}{2c}$ & erigatur ab perpendicularis abſciſſæ in a ita ut $ab = \frac{\sqrt{4ec - dd}}{2c}$) $= c \times \overline{aP^2 + ab^2} = c \times b\text{P}^2$, ſitque PM × P$m = cxx - dx + e$, erit PM × P$m$ ad bP^2 ut c ad unitatem. Itaque in ſectione quavis conica ſi recta AP ſectioni non occurrat, erit, dato angulo APM, rectangulum contentum ſub rectis ad punctum B conſiſtentibus & ad curvam terminatis ad quadratum Fig. 5.

tum diſtantiæ puncti P a puncto dato b in ratione conſtanti, quæ in circulo eſt ratio æqualitatis. Manifeſtum autem eſt eandem methodum adhiberi poſſe lineæ quarti ordinis quam abſciſſa ſecat in duobus tantum punctis, vel lineæ ordinis cujuſcunque quam abſciſſa ſecat in punctis binario paucioribus numero qui figuræ ordinem deſignat.

§ 8. Hiſce præmiſſis, progredimur ad linearum geometricarum proprietates minus obvias exponendas eodem fere ordine quo ſe nobis obtulerunt. Utebamur autem lemmate ſequenti ex fluxionum doctrina petito, quodque in tractatus de hiſce nuper editi Art. 717. demonſtravimus; harum tamen aliquas per algebram vulgarem demonſtrari poſſe poſtea obſervavimus.

Lemma. Si quantitatibus x, y, z, u, &c. ſimul fluentibus, ut & quantitatibus X, Y, Z, V, &c. ſit factum ex prioribus ad factum ex poſterioribus in ratione conſtanti quacunque, erit $\frac{\dot{x}}{x} + \frac{\dot{y}}{y} + \frac{\dot{z}}{z} + \frac{\dot{v}}{v} +$ &c. $= \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} + \frac{\dot{V}}{V} +$ &c. Porro, brevitatis gratia, quantitates appellamus ſibi mutuo *reciprocas*, quarum in ſe mutuo ductarum factum eſt unitas, ſic $\frac{1}{x}$ dicimus *reciprocam* eſſe ipſius x, & $\frac{1}{y}$ ipſius y.

§ 9. Theor. I. *Occurrat recta quævis per punctum datum ducta lineæ geometricæ cujuſcunque ordinis in tot punctis quot ipſa eſt dimenſionum; rectæ figuram in his punctis contingentes abſcindant ab alia rectâ poſitione data per idem punctum datum ductâ ſegmenta totidem hoc puncto terminata; & horum ſegmentorum* reciproca *eandem*

dem ſemper conficient ſummam, modo ſegmenta ad contrarias partes puncti dati ſita contrariis ſignis afficiantur.

Sit P punctum datum, PA & P*a* rectæ quævis duæ ex P ductæ quarum utraque curvam ſecat in tot punctis A, B, C, et *a*, *b*, *c*, &c. quot ipſa eſt dimenſionum. Abſcindant tangentes AK, BL, CM, &c. et *ak*, *bl*, *cm*, &c. a recta EP per punctum datum P ducta ſegmenta PK, PL, PM, &c. et P*k*, P*l*, P*m*, &c. dico fore $\frac{1}{PK}+\frac{1}{PL}+\frac{1}{PM}+$ &c. $=\frac{1}{Pk}+\frac{1}{Pl}+\frac{1}{Pm}+$ &c. atque hanc ſummam manere ſemper eandem manente puncto P & recta PE poſitione data. Fig. 6.

Supponamus enim rectas ABC, *abc* motibus ſibi parallelis deferri, ita ut earum occurſus P progrediatur in recta PE poſitione data; cumque ſit ſemper AP × BP × CP × &c. ad *a*P × *b*P × *c*P in ratione conſtanti per Art. 5. repræſentet $A\dot{P}$ fluxionem ipſius AP, $B\dot{P}$ fluxionem rectæ BP, & $C\dot{P}$, $E\dot{P}$, &c. fluxiones rectarum CP, EP, &c. reſpectivas, ut vitetur inutilis ſymbolorum multiplicatio, eritque (per Art. 8.) $\frac{A\dot{P}}{AP}+\frac{B\dot{P}}{BP}+\frac{C\dot{P}}{CP}+$ &c. $=\frac{a\dot{P}}{aP}+\frac{b\dot{P}}{bP}+\frac{c\dot{P}}{cP}+$ &c. Verum cum recta AP motu ſibi ſemper parallelo deferatur, notiſſimum eſt $A\dot{P}$ fluxionem rectæ AP eſſe ad $E\dot{P}$ fluxionem rectæ EP ut AP ad ſubtangentem PK, adeoque $\frac{A\dot{P}}{AP}=\frac{E\dot{P}}{PK}$. Similiter $\frac{B\dot{P}}{BP}=\frac{E\dot{P}}{PL}$, $\frac{C\dot{P}}{CP}=\frac{E\dot{P}}{PM}$, $\frac{a\dot{P}}{aP}=\frac{E\dot{P}}{Pk}$, $\frac{b\dot{P}}{bP}=\frac{E\dot{P}}{Pl}$ & $\frac{c\dot{P}}{cP}=\frac{E\dot{P}}{Pm}$, unde $\frac{E\dot{P}}{PK}+\frac{E\dot{P}}{PL}$

+

$+ \frac{E\dot{P}}{PM} +$ &c. $= \frac{E\dot{P}}{Pk} + \frac{E\dot{P}}{Pl} + \frac{E\dot{P}}{Pm} +$ &c. et $\frac{1}{PK} + \frac{1}{PL} + \frac{1}{PM} +$ &c. $= \frac{1}{Pk} + \frac{1}{Pl} + \frac{1}{Pm} +$ &c.

Fig. 7.

Hæc ita ſe habent quoties puncta K, L, M, &c. et *k*, *l*, *m*, &c. ſunt omnia ad eaſdem partes puncti P, adeoque fluxiones rectarum AP, BP, CP, &c. *a*P, *b*P, *c*P, &c. omnes ejuſdem ſigni. Si vero, cæteris manentibus, puncta quævis M et *m* cadant ad contrarias partes puncti P, tum creſcentibus reliquis ordinatis AP, BP, &c. neceſſario minuuntur ordinatæ CP & *c*P, earumque fluxiones pro ſubdititiis ſeu negativis habendæ ſunt; adeoque in hoc caſu $\frac{1}{PK} + \frac{1}{PL} - \frac{1}{PM}$, &c. $= \frac{1}{Pk} + \frac{1}{Pl} - \frac{1}{Pm}$, &c. & generaliter in ſummis hiſce colligendis, termini iiſdem vel contrariis ſignis afficiendi ſunt, prout ſegmenta cadunt ad eaſdem vel ad contrarias partes puncti dati P.

§ 10. Si recta PE occurrat curvæ in tot punctis D, E, I, &c. quot ipſa eſt dimenſionum, ſumma $\frac{1}{PK} + \frac{1}{PL} + \frac{1}{PM} +$ &c. quam conſtantem ſeu invariatam manere oſtendimus, æqualis erit ſummæ ſeu aggregato $\frac{1}{PD} + \frac{1}{PE} + \frac{1}{PI} +$ &c. i. e. ſummæ reciprocarum ſegmentis rectæ PE poſitione datæ puncto dato P & curva terminatis; in qua, ſi ſegmentum quodvis ſit ad alteras partes puncti P, hujus reciproca ſubducenda eſt.

§ 11. Si figura ſit ſectio conica, cui recta PE nullibi occurrat, inveniatur punctum b ut in Art. 7. jungatur Pb, huic ducatur ad rectos angulos bd rectam PE ſecans in d, eritque $\frac{1}{PK}+\frac{1}{PL}=\frac{2}{Pd}$. Eſt enim PA × PB ad bP^2 in ratione conſtanti, adeoque (per Art. 8.) $\frac{A\dot{P}}{AP}+\frac{B\dot{P}}{BP}=\frac{2b\dot{P}}{bP}$, unde (quoniam $A\dot{P}$ eſt ad $E\dot{P}$ ut AP ad PK, $B\dot{P}$ ad $E\dot{P}$ ut BP ad PL, & $b\dot{P}$ ad $E\dot{P}$ ut bP ad dP) $\frac{1}{PK}+\frac{1}{PL}=\frac{2}{Pd}$. Fig. 8.

§ 12. Similiter ſi recta EP occurrat lineæ tertii ordinis in unico puncto D, inveniatur punctum b ut in Art. 6, recta bd perpendicularis in junctam bP occurrat rectæ EP in d, & quoniam AP × BP × CP eſt ad DP × bP^2 in ratione conſtanti (*ibid.*) erit $\frac{1}{PK}+\frac{1}{PL}+\frac{1}{PM}=\frac{1}{PD}+\frac{2}{Pd}$. Si autem P$b$ perpendicularis ſit in rectam EP, evaneſcet $\frac{2}{Pd}$. Fig. 9.

§ 13. Aſymptoti linearum geometricarum ex datâ plaga crurum infinitorum per hanc propoſitionem determinantur; eæ enim conſiderari poſſunt tanquam tangentes cruris in infinitum producti. Recta PA aſymptoto parallela curvæ occurrat in punctis A, B, &c. recta autem PE curvam ſecet in D, E, I, &c. ſumatur in hac recta PM ita ut $\frac{1}{PM}$ ſit æqualis exceſſui quo ſumma $\frac{1}{PD}+\frac{1}{PE}+\frac{1}{PI}+$ &c. ſuperat ſummam $\frac{1}{PK}+\frac{1}{PL}+$ &c. & aſymptotos tranſibit per M, ſi vero Fig. 10.

æquales ſint hæ ſummæ, crus curvæ parabolicum erit, aſymptoto abeunte in infinitum.

Fig. 11. § 14. Ad *curvaturam* linearum geometricarum unico theoremate generali definiendam, ſit CDR circulus cui occurrant recta PR in D & R, & recta PC in C & N; ſecet tangens CM rectam PD in M, atque manente recta DR, ſupponamus rectam PCN deferri motu ſibi ſemper parallelo donec coincidant puncta P, D, C, & quæratur ultimus valor differentiæ $\frac{1}{PM}-\frac{1}{PD}$. In recta PN ſumatur punctum quodvis q, occurrat qv parallela tangenti CM rectæ DR in v; ducatur DQ parallela ipſi PN, & QV (parallela rectæ circulum contingenti in D) ſecet DR in V. Erit itaque $\frac{1}{PM}-\frac{1}{PD}=\frac{DM}{PM\times PD}$ (quoniam $DM\times MR=CM^2$) $=\frac{CM^2\times PM}{PM^2\times PD\times MR}$ $=\frac{qv^2\times PM}{Pv^2\times MR\times PM+Pv^2\times MR\times MD}$ (cum $MR\times MD$, ſeu CM^2, ſit ad PM^2 ut qv^2 ad Pv^2) $=\frac{qv^2\times PM}{Pv^2\times MR\times PM+qv^2\times PM^2}=\frac{qv^2}{Pv^2\times MR+qv^2\times PM}$, cujus ultimus valor, evaneſcente PM & coincidentibus qu & Pv cum QV & DV, eſt $\frac{QV^2}{DV^2\times DR}$. Atque idem eſt valor ultimus differentiæ $\frac{1}{PM}-\frac{1}{PD}$ ſi D & C ſint in arcu lineæ cujuſvis ejuſdem curvaturæ cum circulo CDR.

Fig. 12. § 15. Theor. II. *Ex puncto quovis* D *lineæ geometricæ ducantur duæ quævis rectæ* DE, DA, *quarum utraque eam ſecet in tot punctis* D, I, E, *&c.* & D, A, B, *&c. quot ipſa eſt dimenſionum; abſcindant tangentes*

AK, BL, &c. a recta DE ſegmenta DK, DL, &c. Occurrat recta quævis, QV tangenti DT parallela ipſis DA & DE in Q & V, ſitque QV^2 *ad* DV^2 *ut m ad* 1; *ſumatur in DE recta DR ita ut* $\frac{m}{DR}$ *æqualis ſit exceſſui ſummæ* $\frac{1}{DE}+\frac{1}{DI}+$ *&c. ſupra ſummam* $\frac{1}{DK}+\frac{1}{DL}+$ *&c. & circulus ſupra chordam DR deſcriptus rectam DT contingens erit circulus oſculatorius, ſive ejuſdem curvaturæ cum linea geometrica propoſita, ad punctum D.*

Oſtendimus enim in Art. 10. (Fig. 6.) generaliter ſummam $\frac{1}{PK}+\frac{1}{PL}+\frac{1}{PM}+\&c. = \frac{1}{PD}+\frac{1}{PE}+\frac{1}{PI}+\&c.$ & in Art. præcedente invenimus valorem ultimum differentiæ $\frac{1}{PM}-\frac{1}{PD}$, coincidentibus punctis P, D & C, eſſe $\frac{QV^2}{DV^2\times DR}=\frac{m}{DR}$ ſi circulus ejuſdem curvaturæ cum linea geometrica ad punctum D rectæ DE occurrat in R. Unde ſequitur fore $\frac{m}{DR}=\frac{1}{DE}+\frac{1}{DI}+\&c.-\frac{1}{DK}-\frac{1}{DL}-\&c.$ ſive reciprocam ipſi $\frac{1}{m}\times DR$ eſſe æqualem exceſſui quo ſumma reciprocarum ſegmentis puncto D & curva terminatis ſuperat ſummam reciprocarum ſegmentis eodem puncto & tangentibus AK, BL, &c. terminatis. Quoties autem exceſſus hic evadit negativus, chorda DR ſumenda eſt ad alteras partes puncti D, ſemperque adhibenda eſt regula ſuperius deſcripta pro ſignis terminorum dignoſcendis. Si recta DA biſecet angulum EDT recta DE & tangente DT contentum, theorema ſit paulo ſimplicius. Hoc enim in caſu QV = DV, $m = 1$, & $\frac{1}{DR}$ æqualis exceſſui quo $\frac{1}{DE} + \frac{1}{DI} +$ &c. ſuperat $\frac{1}{DK} + \frac{1}{DL} +$ &c.

§ 16. Ex eodem principio conſequitur theorema generale quo determinatur variatio curvaturæ vel menſuræ anguli contactus curva & circulo oſculatorio contenti, in linea quavis geometrica; præmittenda tamen eſt explicatio brevis variationis curvaturæ, cum hæc non ſatis dilucide apud auctores deſcripta ſit. Linea quævis curva a tangente flectitur per curvaturam ſuam, cujus eadem eſt menſura ac anguli contactus curva & tangente contenti; & ſimiliter curva a circulo oſculatorio inflectitur per variationem curvaturæ ſuæ, cujus variationis eadem eſt menſura ac anguli contactus curva & circulo oſculatorio comprehenſi. Occurrat recta TE tangenti DT perpendicularis curvæ in E & circulo oſculatorio in r, & variatio curvaturæ erit ultimo ut Er ſubtenſa anguli contactus EDr ſi detur DT; cumque dato angulo contactus EDr ſit Er ultimo ut DT^3, ut ex Art. 369. tractatus de fluxionibus colligitur, generaliter curvaturæ variatio erit ultimo ut $\frac{Er}{DT^3}$. Utimur circulo ad curvaturam aliarum figurarum definiendam; verum ad variationem curvaturæ menſurandam, quæ in circulo nulla eſt, adhibenda eſt parabola vel ſectio aliqua conica. Quemadmodum autem ex circulis numero indefinitis qui curvam datam in puncto dato contingere poſſunt, unicus dicitur oſculatorius qui curvam adeo intime tangit ut nullus alius circulis inter hunc & curvam duci poſſit; ſimiliter omnium parabolarum quæ eandem habent curvaturam cum linea propoſita ad punctum datum Fig. 13.

(ſunt autem hæ quoque numero infinitæ) ea eandem ſimul habet curvaturæ variationem, quæ, non ſolum arcum curvæ tangit & oſculat, ſed adeo premit ut nullus alius arcus parabolicus duci poſſit inter eas, reliquis omnibus arcubus parabolicis tranſeuntibus vel extra vel intra utraſque. Qua vero ratione hæc parabola determinari poſſit, ex iis quæ alibi fuſius explicavimus facile intelligitur.

Sit DE arcus curvæ, DT tangens, TEK recta tangenti perpendicularis, ſitque rectangulum ET × TK ſemper æquale quadrato tangentis DT, & curva SKF locus puncti K, qui rectæ DS curvæ normali occurrat in S, quemque tangat in S recta SV tangentem TD ſecans in V. Recta DS erit diameter circuli oſculatorii, & biſecta DS in *ſ*, erit *ſ* centrum curvaturæ; junctâ autem V*ſ*, ſi angulus SDN conſtituatur æqualis angulo *ſ*VD ex altera parte rectæ DS, & recta DN circulo oſculatorio occurrat in N; tum parabola diametro & parametro DN deſcripta, quæque rectam DT contingit in D, ipſa erit cujus contactus cum linea propoſita in D intimus erit atque maxime perfectus ſeu proximus. Omnes autem parabolæ aliâ quavis chorda circuli oſculatorii tanquam diametro & parametro deſcriptæ, & rectam DT contingentes in D, eandem habent curvaturam cum linea propoſita in puncto D. Qualitas curvaturæ a *Newtono* in opere poſthumo nuper edito explicata eſt potius variatio radii curvaturæ; eſt enim ut fluxio radii curvaturæ applicata ad fluxionem curvæ, vel (ſi R denotet radium circuli oſculatorii & S arcum curvæ ut $\frac{\dot{R}}{\dot{S}}$. Ipſa autem curvatura eſt inverſe ut radius R, & variatio curvaturæ ut $\frac{-\dot{R}}{RR\dot{S}}$, quæ eſt menſura anguli contactus curvâ & circulo oſculatorio contenti. Harum autem una ex alterâ datâ facile derivatur. Variatio radii curvaturæ in curvâ quavis DE eſt ut tangens anguli DVS vel DV*ſ*, & in parabolâ quavis eſt ſemper ut tangens anguli contenti diametro per punctum contactus tranſeunte & rectâ ad curvam perpendiculari. Hæc ex theoremate ſequenti generali deduci poſſunt.

Fig. 14. § 17. Theor. III. *Sit* D *punctum in linea quavis geometrica datum, occurrat* DS *diameter circuli oſculatorii per* D *ducta curvæ in tot punctis* D, A, B, *&c. quot ipſa eſt dimenſionum; ducatur* DT *curvam contingens in* D, *quæ curvam ſecet in punctis* I, *&c. binario paucioribus, & occurrat tangentibus* AK, BL, *&c. in* K, L, *&c. eritquè variatio curvaturæ, ſive menſura anguli contactus curva & circulo oſculatorio comprehenſi, directe ut exceſſus quo ſumma reciprocarum ſegmentis tangentis* DT *puncto contactus* D *& tangentibus* AK, BL, *&c. terminatis ſuperat ſummam reciprocarum ſegmentis eodem puncto & curva terminatis, & inverſe ut radius curvaturæ*, i. e. *ut* $\frac{1}{DS} \times \overline{\frac{1}{DK} + \frac{1}{DL} + \&c. - \frac{1}{DI} - \&c.}$

Ducatur enim recta D*k* curvam ſecans in *e*, *i*, &c. circulum oſculatorium in R; ſitque angulus *k*DT quam minimus; hujus ſupplementum ad duos rectos biſecetur recta *Dab*, quæ lineæ geometricæ propoſitæ occurrat in punctis D, *a*, *b*, &c. & ductæ tangentes *ak*, *bl*, &c. ſecent rectam D*k* in punctis *k*, *l*, &c. eritque per propoſitionem præcedentem $\frac{1}{DR} = \frac{1}{De} + \frac{1}{Di} - \frac{1}{Dk} - \frac{1}{Dl}$, &c. Unde $\frac{1}{DR} - \frac{1}{De}$ (ſive $\frac{Re}{DR \times De}$) $= \frac{1}{Di} -$

$-\frac{1}{Dk}-\frac{1}{Dl}-$ &c. Proinde coincidentibus rectis Dk & DK, ſeu evaneſcente angulo kDK, erit ultimo $\frac{Re}{DR\times De}$ æqualis $\frac{1}{DI}-\frac{1}{DK}-\frac{1}{DL}-$ &c. Sit erT perpendicularis tangenti in T, atque occurrat circulo oſculatorio in r; cumque ſit re ultimo ad Re ut eT ad De, erit ultimo $\frac{Re}{DR\times De}=\frac{re}{DR\times eT}=\frac{re\times DS}{DR\times DT^2}$ ſive $\frac{re\times DS}{DT^3}$. Menſura autem anguli contactus rDe curva & circulo oſculatorio contenti, ſive variatio curvaturæ, eſt ut $\frac{re}{DT^3}$ adeoque ut $\frac{1}{DS}\times\overline{\frac{1}{DI}-\frac{1}{DK}-\frac{1}{DL}}$, &c.

§ 18. Variatio autem radii curvaturæ, ſive hujus qualitas a *Newtono* deſcripta, ex priori facillime colligitur. Junctis enim SI, SK, SL, &c. erit hæc variatio radii oſculatorii ut exceſſus quo ſumma tangentium angulorum DKS, DLS, &c. ſuperat ſummam tangentium angulorum DIS, &c. Creſcit autem curvatura a puncto D verſus e, & minuitur radius oſculatorius, quoties arcus De tangit circulum oſculatorium DR interne, vel cum $\frac{1}{DK}+\frac{1}{DL}+$ &c. ſuperat $\frac{1}{DI}+$ &c. at contrà minuitur curvatura a D verſus e, & augetur radius circuli oſculatorii, quoties arcus curvæ De tangit arcum circularem externe vel tranſit intra circulum & tangentem adeoque cum DR ſit ultimo minor quam De vel cum $\frac{1}{DI}+$ &c. ſuperat $\frac{1}{DK}+\frac{1}{DL}+$ &c.

§ 19. Sumatur igitur in tangente DT recta DV ita ut $\frac{1}{DV}=\frac{1}{DK}+\frac{1}{DL}+$ &c. $-\frac{1}{DI}-$ &c. jungatur ſV, conſtituatur angulus SDN æqualis DVſ, atque occurrat recta DN circulo oſculatorio in N; & parabola diametro DN deſcripta, cujus parameter eſt DN, quæque rectam DT contingit in D, eandem habebit variationem curvaturæ cum lineâ geometricâ propoſita in puncto D. Ex iiſdem principiis alia quoque theoremata deducuntur, quibus variatio curvaturæ in lineis geometricis generaliter definitur.

§ 20. Ut hæc theoremata ad formam magis geometricam reducantur, lemmata quædam ſunt præmittenda, quibus doctrina de diviſione rectarum harmonicâ Fig. 15. amplior & generalior reddatur. In recta quavis DI, ſumptis æqualibus ſegmentis DF & FG, ducantur a puncto quovis V quod non eſt in rectâ DI tres rectæ VD, VF, VG, & quarta VL ipſi DI parallela, atque hæ quatuor rectæ, a Cl. D. *De la Hire*, Harmonicales dicuntur. Recta vero quævis, quæ quatuor harmonicalibus occurrit ab iiſdem harmonice ſecatur. Occurrat recta DC harmonicalibus VD, VF, VG, & VL in punctis D, A, B, C; eritque DA ad DC ut AB ad BC. Ducatur enim per punctum A recta MAN ipſi DI parallela, quæ occurrat rectis VD & VG in M & N; & ob æquales DF & FG, æquales erunt MA & AN. Eſt autem DA ad DC ut AM (ſive AN) ad VC, adeoque ut AB ad BC. Manifeſtum eſt rectam, quæ uni harmonicalium parallela eſt, dividi in æqualia ſegmenta a tribus reliquis. Occurrat recta BH parallela ipſi VF reliquis VG, VC, VD in B, K, & H; eritque VK ad KB ut FG (vel DF) ad VF adeoque ut VK ad KH, & proinde BK=KH.

§ 21. Hinc ſequitur, ſi recta quævis a quatuor rectis ab eodem puncto ductis ſecetur harmonice, aliam quamvis rectam quæ his quatuor rectis occurrit harmonice ſecari ab iiſdem; eam vero quæ parallela eſt uni quatuor rectarum in ſegmenta æqualia dividi a tribus reliquis. Sit DA ad DC ut AB ad BC, jungantur VA, VB, VC, & VD; occurrant rectæ MAN, DFG ipſi VC parallelæ rectis VD, VA, & VB in M, A, N & D, F, G; eritque MA ad VC ut DA ad DC, vel AB ad BC, adeoque ut AN ad VC; MA = AN, & DF = FG; &, per præcedentem, recta quævis quæ ipſis VD, VA, VB, VC occurrit harmonice ſecabitur ab iiſdem.

Fig. 16. n. 1.

§ 22. Ex puncto D ducantur duæ rectæ DAC, *Dac* rectas VA & VC ſecantes in punctis A, C atque *a*, *c*; junctæ A*c* & *a*C ſibi mutuo occurrant in Q, & ducta VQ harmonice ſecabit rectam DAC vel aliam quamvis rectam ex puncto D ad eaſdem rectas ductam. Secet enim VQ rectam AC in B, & per punctum Q ducatur recta MQN parallela ipſi DC, quæ occurrat rectis D*a*, VA & VC in punctis M, R, & N; cumque ſit MR ad MQ ut DA ad DC, & MQ ad MN in eadem ratione, erit quoque RQ ad QN ut DA ad DC. Sed RQ eſt ad QN ut AB ad BC. Quare DA eſt ad DC ut AB ad BC. Hæc eſt Prop. 20ma, Lib. I. ſectionum conicarum Cl. *De la Hire*.

§ 23. Sit DA ad DC ut AB ad BC, eritque $\frac{2}{DB}$ æqualis ſummæ vel differentiæ ipſarum $\frac{1}{DA}$ & $\frac{1}{DC}$ prout puncta A & C ſunt ad eaſdem vel contrarias partes puncti D. Sint imprimis puncta A & C ad eaſdem partes puncti D, cumque ſit $DA \times BC = DC \times AB$, i. e. $DA \times \overline{DC - DB} = DC \times \overline{DB - DA}$, vel $DA \times \overline{DB - DC} = DC \times \overline{DA - DB}$ erit $2DA \times DC = DA \times DB +$

DC × DB, adeoque $\frac{2}{DB} = \frac{1}{DA} + \frac{1}{DC}$. Sint nunc n. 2 & 3. puncta A & C ad contrarias partes puncti D, eritque vel $DA \times \overline{DB - DC} = DC \times \overline{DB + DA}$, vel $DA \times \overline{DB + DC} = DC \times \overline{DB - DA}$, adeoque $\frac{2}{DB} = \frac{1}{DC} - \frac{1}{DA}$ cum puncta B & C ſunt ad eaſdem partes puncti D, vel $\frac{2}{DB} = \frac{1}{DA} - \frac{1}{DC}$ quoties puncta A & B ſunt ad eaſdem partes puncti D. Si igitur, datis puncto D & rectis VF & VC poſitione, ducatur ex puncto D recta quævis illis occurrens in punctis A & C, & in eadem recta ſumatur ſemper DB ita ut $\frac{2}{DB} = \mp \frac{1}{DA} \mp \frac{1}{DC}$, ubi ſupponitur terminos $\frac{1}{DA}$ & $\frac{1}{DC}$ iiſdem vel contrariis ſignis afficiendos eſſe prout puncta A & C ſunt ad eaſdem vel contrarias partes puncti D, erit locus puncti B ipſa harmonicalis VG quæ rectam DFG rectæ VC parallelam ſecat in G ita FG = DF; quæque tranſit per punctum Q ubi (ductâ D*ac* quæ iiſdem rectis VF & VC occurrat in *a* et *c*) junctæ A*c* et *a*C ſe mutuo decuſſant.

Fig. 17. § 24. Si in recta DA ſumatur ſemper D*b* ita ut $\frac{1}{Db} = \frac{1}{DA} \mp \frac{1}{DC}$; ducatur DF parallela rectæ VC quæ rectæ VF occurrat in F, & DH parallela rectæ VF quæ rectæ VC occurrat in H, & ducta diagonalis HF erit locus puncti *b*; nam ex hypotheſi $\frac{1}{Db} = \frac{2}{DB}$, & DB = 2D*b*; adeoque cum VG ſit locus puncti B erit punctum *b* ad rectam HF, ſi puncta A & C ſint ad eaſdem partes puncti D. Si autem ſupponatur $\frac{1}{Db} = \frac{1}{DA} + \frac{1}{DC}$, eadem conſtructio inſerviet pro determinando puncto *b*, ſi ſubſtituatur loco rectæ VC alia *vc* rectæ VC parallela ad æqualem diſtantiam a puncto D ſed ad contrarias partes.

§ 25. Ex puncto dato D ducatur recta quævis DM quæ tribus rectis poſitione datis occurrat in punctis A, C, E; & ſumatur ſemper DM ita ut $\frac{1}{DM} = \frac{1}{DA} + \frac{1}{DC} + \frac{1}{DE}$ (ubi termini ſunt contrariis ſignis afficiendi quoties rectæ DA, DC vel DE ſunt ad contrarias partes puncti D); ſupponatur $\frac{1}{DA} + \frac{1}{DC} = \frac{1}{DL}$, eritque L ad rectam poſitione datam per præcedentem; adeoque, cum ſit $\frac{1}{DM} = \frac{1}{DL} + \frac{1}{DE}$, erit punctum M ad poſitione datam, per eandem. Compoſitio autem problematis facile ex dictis perficitur. Sint VA, VC & *v*E tres rectæ poſitione datæ, & compleatur parallelogrammum DFVH, ducendo DF & DH rectis VC & VF reſpective parallelas, & occurrat recta *v*E diagonali in *v*; deinde compleatur parallelogrammum D*fvh* ducendo rectas D*f* & D*h* rectis *v*E & HF parallelas quæ rectis HE & *v*E occurrant in punctis *f* & *h*; & diagonalis *hf* erit locus puncti M. Occurrat enim recta DA rectis HF & *hf* in L & M; eritque, ex præcedentibus, $\frac{1}{DM} = \frac{1}{DL} + \frac{1}{DE} = \frac{1}{DA} + \frac{1}{DC} + \frac{1}{DE}$. Alia conſtructio ex Art. 22. deducitur.

§ 26.

§ 26. Recta quævis ex puncto dato D ducta occurrat rectis poſitione datis in punctis A, B, C, E, &c. et in hac recta ſumatur ſemper $\frac{1}{DM} = \frac{1}{DA} \mp \frac{1}{DB} \mp \frac{1}{DC} \mp \frac{1}{DE}$, &c. eritque locus puncti M ſemper ad rectam poſitione datam. Demonſtratur ad modum præcedentis.

Fig. 18. § 27. Theor. IV. *Circa datum punctum* P *revolvatur recta* PD *quæ occurrat lineæ geometricæ cujuſcunque ordinis in tot punctis* D, E, I, *&c. quot ipſa eſt dimenſionum, & ſi in eadem recta ſumatur ſemper* PM *ita ut* $\frac{1}{PM} = \frac{1}{PD} \mp \frac{1}{PE} \mp \frac{1}{PI} \mp$ *&c. (ubi ſigna terminorum regulam ſæpius deſcriptam obſervare ſupponimus) erit locus puncti* M *linea recta.*

Ducatur enim ex polo P recta quævis poſitione data PA, quæ curvæ occurrat in tot punctis A, B, C, &c. quot ipſa eſt dimenſionum. Ducantur rectæ AK, BL, CN curvam in his punctis contingentes, quæ occurrant rectæ PD in totidem punctis K, L, N, &c. et per Art. 10. $\frac{1}{PD} \mp \frac{1}{PE} \mp \frac{1}{PI} \mp$ &c. $= \frac{1}{PK} \mp \frac{1}{PL} \mp \frac{1}{PN} \mp$ &c.. Unde $\frac{1}{PM}$ æqualis eſt huic ſummæ, cumque poſitione detur recta PA, & maneant rectæ AK, BL, CN, &c. dum recta PD circa polum P revolvitur, erit punctum M ad lineam rectam, per articulum præcedentem; quæ per ſuperius oſtenſa ex datis tangentibus AK, BL, &c. determinari poteſt.

§ 28. Sicut recta P*m* medium eſt harmonicum inter duas rectas PD & PE, cum $\frac{2}{Pm} = \frac{1}{PD} + \frac{1}{PE}$; ſimiliter Pm dicatur *medium* harmonicum inter rectas quaſlibet PD, PE, PI, &c. quarum numerus eſt n, cum $\frac{n}{Pm} = \frac{1}{PD} \mp \frac{1}{PE} \mp \frac{1}{PI} \mp$ &c. Et ſi ex puncto dato P recta quævis ducta lineam geometricam ſecet in tot punctis quot ipſa eſt dimenſionum, in quâ ſumatur ſemper Pm medium harmonicum inter ſegmenta omnia ductæ ad punctum datum P & curvam terminata, erit punctum m ad rectam lineam. Erit enim $\frac{1}{PM} = \frac{n}{Pm}$ adeoque Pm ad PM ut n ad unitatem; cumque punctum M ſit ad rectam lineam, per præcedentem, erit m quoque ad rectam lineam. Atque hoc eſt theorema *Coteſii*, vel eidem affine.

§ 29. Sint a, b, c, d, &c. radices æquationis ordinis n, V ultimus ejus terminus quem ordinata ſeu radix y non ingreditur, P coefficiens termini penultimi, M medium harmonicum inter omnes radices, ſeu $\frac{n}{M} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} +$ &c. Cum igitur ſit V factum ex radicibus omnibus a, b, c, &c. in ſe mutuo ductis, ſitque P ſumma factorum cum radices omnes unâ dempta in ſe mutuo ducuntur, erit $P = \frac{V}{a} + \frac{V}{b} + \frac{V}{c} + \frac{V}{d} +$ &c. $= \frac{nV}{M}$, adeoque $M = \frac{nV}{P}$. Sic, ſi æquatio ſit quadratica, cujus radices duæ ſint a et b, erit $M = \frac{2ab}{a+b}$ (aſſumptâ æquatione generali ſectionum conicarum Art. 1. propoſitâ) $= \frac{2cxx - 2dx + 2e}{ax - b}$. In æquatione cubica cujus tres radices ſunt a, b, c, erit $M = \frac{3abc}{ab + ac + bc}$ (ſi aſſumatur æquatio generalis linearum tertii ordinis ibidem propoſiti) $= \frac{3fx^3 - 3gx^2 + 3hx - 3k}{cxx - dx + e}$.

Fig. 19. § 30. Occurrant rectæ quævis duæ Pm & Pμ, ex puncto P ductæ, lineæ geometricæ in punctis D, E, I, &c. et d, e, i, &c. ſitque Pm medium harmonicum inter ſegmenta prioris ad punctum P & curvam terminata, & Pμ medium harmonicum inter ſegmenta ſimilia poſterioris rectæ; juncta μm occurrat abſciſſæ AP in H, eritque $\text{PH} = \frac{n\text{V}\dot{x}}{\dot{\text{V}}}$ vel PH ad Pm ut P ad $\frac{\dot{\text{V}}}{\dot{x}}$. Secet enim abſciſſa curvam in tot punctis B, C, F, &c. quot ipſa eſt dimenſionum; cumque ultimus terminus æquationis (i. e. V) ſit ad BP × CP × FP × &c. in ratione conſtanti, ut ſupra (Art. 5.) oſtendimus, erit (per Art. 8. $\frac{\dot{\text{V}}}{\text{V}} = \frac{\dot{x}}{\text{BP}} \mp \frac{\dot{x}}{\text{CP}} \mp \frac{\dot{x}}{\text{FP}} \mp$ &c. adeoque $\frac{n}{\text{PH}} = \frac{1}{\text{BP}} \mp \frac{1}{\text{CP}} \mp \frac{1}{\text{FP}} \mp$ &c. $= \frac{\dot{\text{V}}}{\text{V}\dot{x}}$, & $\text{PH} = \frac{n\text{V}\dot{x}}{\dot{\text{V}}}$ (quoniam recta $\text{PM} = \frac{n\text{V}}{\text{P}}$) $= \text{P}m \times \frac{\text{P}\dot{x}}{\dot{\text{V}}}$. In ſectionibus conicis eſt PH ad Pm ut $ax - b$ ad $2cx - d$; & in lineis tertii ordinis ut $cxx - dx + e$ ad $3fxx - 2gx + h$.

§ 31. Si deſideretur propoſitionis præcedentis demonſtratio ex principiis pure algebraicis petita, ea ope ſequentis *Lemmatis* perfici poterit. Sit abſciſſa AP $= x$, ordinata PD $= y$, ultimus terminus æquationis lineam geometricam definientis $\text{V} = \text{A}x^n + \text{B}x^{n-1} + \text{C}x^{n-2}$ + &c. penultimi coefficiens $\text{P} = ax^{n-1} + bx^{n-2} + cx^{n-3}$ + &c. et ſit Q quantitas quæ formatur ducendo terminum quemque quantitatis V in indicem ipſius x in hoc termino & dividendo per x, i. e. ſit $Q = nAx^{n-1} + \overline{n-1} \times Bx^{n-2} + \overline{n-2} \times Cx^{n-3}$ + &c. (quæ ipſa eſt quantitas quam $\frac{\dot{V}}{\dot{x}}$ dicimus). Ducatur ordinata Dp quæ angulum quemvis datum ApD cum abſciſſa conſtituat, ſintque rectæ PD, pD et Pp ut datæ l, r et k; dicatur $pD = u$, $Ap = z$, & tranſmutetur æquatio propoſita ad ordinatam u & abſciſſam z; & æquationis novæ, cum ſit $z = AP$, terminus ultimus v erit æqualis ipſi V, penultimi autem coefficiens p erit æqualis $\pm \frac{Qk+Pl}{r}$.

Cum enim ſit PD $(=y)$ ad pD $(=u)$ ut l ad r, erit $y = \frac{lu}{r}$; ſit autem Pp ad pD $(=u)$ ut k ad r, erit $Pp = \frac{ku}{r}$, & $AP = x = Ap \pm Pp = z \pm \frac{ku}{r}$. His autem valoribus pro y et x ſubſtitutis in æquatione propoſita lineæ geometricæ, prodibit æquatio relationem co-ordinatarum z et u definiens. Ad hujus ultimum terminum v & penultimum pu determinandum, ſufficit hos valores ſubſtituere in ultimo V, & penultimo Py, æquationis propoſitæ, atque terminos reſultantes colligere in quibus ordinata u vel non reperitur, vel unius tantum dimenſionis; horum enim ſumma dat pu, illorum v. Subſtituatur pro x ipſius valor $z \pm \frac{ku}{r}$ in quantitate V vel $Ax^n + Bx^{n-1} + Cx^{n-2}$ + &c. et termini reſultantes $Az^n \pm Bz\frac{nAz^{n-1}ku}{r} + Bz^{n-1} \pm \overline{n-1} \times \frac{Bz^{n-2}ku}{r} + Cz^{n-2} \pm \overline{n-2} \times \frac{Cz^{n-3}ku}{r} +$ &c. ſoli ad rem faciunt de qua nunc agitur. Subſtituatur deinde pro x idem valor, & pro y ipſius valor $\frac{lu}{r}$, in quantitate $Py = \overline{ax^{n-1} + bx^{n-2} + cx^{n-3} + \&c.} \times y$; & termini reſultantes ſoli $\overline{az^{n-1}} + \overline{bz^{n-2}} + \overline{cz^{n-3}} + \&c. \times \frac{lu}{r}$ ſunt nobis retinendi. Supponatur nunc $z = x$, & ſumma priorum ſit æqualis $V \pm \frac{Qku}{r}$, & poſteriorum ſumma $= \frac{Plu}{r}$. Unde manifeſtum eſt ultimum terminum æquationis novæ $v = V$, & penultimum $pu = \frac{Pl \pm Qk}{r} \times u$.

§ 32. Sit nunc P*m* medium harmonicum inter ſegmenta PD, PE, PI, &c. et Pμ medium harmonicum inter ſegmenta P*d*, P*e*, P*i*, &c. ut in Art. 30. juncta μm ſecet abſciſſam in H; & ſupponamus Pμ ordinatæ *p*D parallelam eſſe. Ducatur μs abſciſſæ parallela, quæ rectæ P*m* occurrat in *s*; eritque P*s* ad Pμ ut PD ad *p*D vel ut *l* ad *r*, et μs ad Pμ ut *k* ad *r*. Cumque ſit $P\mu = \frac{nv}{p}$ (per Articulum præcedentem) $\frac{nVr}{Pl \pm Qk}$, erit $ms = Pm \pm Ps = \frac{nV}{P} \pm \frac{nvl}{pr} = \frac{nV}{P} \pm \frac{nVl}{Pl + Qk} = \frac{nVQk}{P \times \overline{Pl \pm Qk}}$. Eſt autem *ms* ad $s\mu$ ut P*m* ad PH, i. e. $\frac{nVQk}{P \times \overline{Pl \pm Qk}}$ ad $\frac{nVk}{Pl \pm Qk}$ ut P*m* ad PH; adeoque Q ad P ut P*m* ad PH, vel $PH = Pm \times \frac{P}{Q}$ vel $\frac{nV}{Q}$. Cum igitur valor rectæ PH non pendeat a quantitatibus *l*, *k* et *r*; ſed, his mutatis, ſit ſemper idem, erit punctum μ ad rectam poſitione datam, ut in Theor. 4. aliter oſtendimus. Quin & valor rectæ PH is eſt quem in Art. 29. alia methodo definivimus; & recta H*m* omnes

nes rectas ex P ductas ſecat harmonice, ſecundum definitionem ſectionis harmonicæ in Art. 28. generaliter propoſitam.

SECTIO II.

De Lineis ſecundi ordinis, ſive ſectionibus conicis.

§ 33. EX iis quæ generaliter de lineis geometricis in ſectione primâ demonſtrata ſunt, ſponte fluunt proprietates linearum ſecundi, tertii, & ſuperiorum ordinum. Quæ ad ſectiones conicas ſpectant optime derivantur ex proprietatibus circuli, quæ figura baſis eſt coni. Verum ut uſus theorematum præcedentium clarius pateat, & figurarum analogia illuſtretur, operæ pretium erit harum quoque affectiones ex præmiſſis deducere. Doctrina autem conica de diametris, earumque ordinatis (quibus parallelæ ſunt rectæ ſectionem contingentes ad vertices diametri) & de parallelarum ſegmentis quæ rectis quibuſcunque occurrunt, & aſymptotis, tota facillime fluit ex iis quæ Art. 4. & 5. oſtenſa ſunt.

§ 34. Rectæ AB & FG ſectione conicæ inſcriptæ occurrant ſibi mutuo in puncto P; ductæ AK, BL, FM, GN ſectionem contingentes occurrant rectæ PE, per P ductæ in punctis K, L, M, N; eritque ſemper $\frac{1}{PK} \pm \frac{1}{PL} = \frac{1}{PM} \mp \frac{1}{PN}$ (ſi recta PE curvæ occurrat in punctis D & E) $= \frac{1}{PD} \mp \frac{1}{PE}$. Segmentis au- Fig. 20.

tem quæ ſunt ad eaſdem partes puncti P eadem præponuntur
ſigna; iiſque quæ ſunt ad oppoſitas partes
Fig. 21. puncti P ſigna præponuntur contraria. Hinc ſi biſecetur
DE in P, & ex puncto P ducatur recta quævis
ſectionem ſecans in punctis A et B, unde ducantur rectæ
AK et BL curvam contingentis quæ rectam DE
Fig. 22. ſecent in K et L; erit ſemper PK = PL. Quod ſi
DE ſectioni non occurrat, ſitque P punctum ubi diameter
quæ biſecat rectas ipſi DE parallelas eidem
occurrit; erit in hoc quoque caſu PK = PL.

Fig. 23. § 35. Concurrant rectæ AB et FG ſectioni conicæ incriptæ in puncto P; ducantur rectæ ſectionem contingentes in punctis A et F quæ ſibi mutuo occurrant in K, & juncta PK tranſibit per occurſum rectarum quæ ſectionem contingunt in punctis B et G. Si enim recta PK non tranſeat per occurſum rectarum ſectionem tangentium in B et G, huic occurrat in N illi in L; cumque $\frac{1}{PK} \mp \frac{1}{PL} = \frac{1}{PK} \mp \frac{1}{PN}$ per præcedentem, erit PL = PN; & coincidunt puncta L et N contra hypotheſin.

§ 36. Eadem ratione patet rectas AG et BF ſibi mutuo occurrere in π puncto rectæ LK; adeoque puncta P, K, π, L eſſe in eadem recta linea. Hinc datis tribus punctis contactus A, B, et F, cum duabus tangentibus AK et FK, ſectio conica facile deſcribitur. Revolvatur enim recta KπP circa tangentium occurſum K ut polum, quæ occurrat rectis AB et FB in punctis P et π; & junctæ Aπ, FP occurſo ſuo G deſcribent ſectionem conicam quæ tranſibit per tria puncta data A, B, F & continget rectas AK et FK in A et F.

§ 37. Cæteris manentibus, occurrant rectæ AF et BG ſibi mutuo in puncto p, tangentes AK et BL in R, atque tangentes FK et GL in Q; & puncta R, π, Q et p erunt in eadem recta linea; ſimiliter occurrant tangentes AK et GQ in m; tangentes BR et FK in n; & puncta P; m, n, p erunt in eadem recta linea. Demonſtratur ad modum Art. 35. Fig. 24.

§ 38. Hinc datis quatuor punctis contactus A, B, F, G cum unica tangente AK, occurſus rectarum AB et FG, AF et BG, atque AG et BF, dabunt puncta P, p, et π; junctæ autem Pp, Pπ, et $p\pi$ ſecabunt tangentem datam AK in tribus punctis m, K et R unde ductæ mG, KF, RB ſectionem conicam contingent in punctis datis G, F et B.

§ 39. Datis quatuor tangentibus RK, KQ, QL, LR et unico puncto contactus A, occurſus tangentium RK et LQ, LR et QK dabunt puncta m et n. Jungantur LK et nm; & occurſus rectarum LK et RQ, LK et nm, RQ et nm, dabunt puncta π, P et p; junctæ vero PA, πA et pA ſecabunt tangentes RL, QK et QL in punctis contactus B, G et F.

§ 40. Datis quinque punctis contactus A, B, F, G, et f, junctæ GF et Gf rectæ AB occurrant in punctis P et X; junctæ AF et Af occurrant rectæ BG in p et x; & junctæ Pp, Xx occurſu ſuo dabunt punctum m; unde ductæ mA et mG ſectionem conicam tangent in A et G; & ſimiliter determinantur rectæ quæ curvam contingent in punctis reliquis B, F et f.

§ 41. Dentur quinque rectæ ſectionem conicam contingentes, VK, KQ, QL, Lu, et uV; occurſus tangentium VK et LQ dabit punctum m; occurſus tangentium

KQ et Lu dabit punctum n; jungantur mn, LK, VL et mu; recta LK ſecabit rectam mn in P; & recta LV ſecabit ipſam mu in X; juncta autem PX ſecabit tangentes VK et uL in punctis contactus A et B. Similiter reliqua puncta contactus determinantur.

Fig. 25. § 42. Datis tribus tangentibus AK, BK, et RL, cum duobus punctis contactus A et B, facillime determinatur tertium, per Art. 35. Occurrat enim tangens RL reliquis tangentibus in R et L, atque junctæ AL et BR, ſe mutuo decuſſent in π, juncta Kπ ſecabit tangentem RL in tertio puncto contactus F; & ſectio conica deſcribi poteſt ut in Art. 36.

Fig. 26. § 43. Dentur quatuor tangentes KQ, QL, LR, et RK cum unico puncto D ſectionis conicæ quod non ſit in aliquâ quatuor tangentium. Inveniantur puncta P, p et π ut in Art. 39. Jungantur PD, pD, et πD; & ducta PZ rectæ pD parallela occurrat rectæ RQ in Z; & bifariam ſecetur PZ in S; & ducta pS ſecabit rectam PD in E puncto curvæ; vel occurrat PD rectæ RQ in z, et (per Art. 23.) ſecetur PD harmonice in z et E. Ducta autem Dπ ſecabit junctam pE in e, et Eπ ſecabit ipſam pD in d, ita ut hæc quoque puncta d, e ſint ad curvam.

Fig. 27. n. 1. § 44. Ex puncto K ducantur duæ tangentes ad ſectionem conicam in A et B; ex puncto A ducantur rectæ duæ AF et AG ſectioni occurrentes in F et G; juncta BG ſecet AF in P, et juncta BF ſecet rectam AG in π; eruntque puncta P, K, π in eadem recta linea, per Art. 36.

n. 2. Verum propoſitio hæc generalior eſt. Si enim a puncto quovis K ducantur duæ rectæ K Aa, KBb ſectionem ſecantes in punctis A, *a* et B, *b*; et ex punctis A et *a* ducantur rectæ & ſectionem AF et *a*G; juncta autem BF ſecet *a*G in P, & ducta *b*G ſecet AF in π; erunt puncta P, K, π in eadem recta linea; quod variis modis aliàs demonſtravimus, unde expeditam methodum olim deduximus ſectionem conicam deſcribendi per data quævis quinque puncta. Sint A, *a*, B, *b* et F puncta quinque data; concurrant rectæ A*a* et B*b* in K; jungantur AF et BF; revolvatur recta PKπ circa polum K, quæ occurrat his rectis in π et P; et ductæ *a*P, *b*π concurſu ſuo G ſectionem deſcribent.

§ 45. Sit P punctum datum extra ſectionem conicam, unde ducta quævis ſectioni occurrat in D et E; et ſi $\frac{2}{PM} = \frac{1}{PD} \mp \frac{1}{PE}$ erit M ad lineam rectam AB quæ ſectioni occurrit in punctis A et B, ita ut ductæ PA et PB, erunt contingentes ſectionis. Si vero punctum *p* ſit in medio puncto rectæ AB intra ſectionem, ſitque $\frac{2}{pm} = \frac{1}{pd} \mp \frac{1}{pe}$, locus puncti *m* erit recta *ab* per P ducta ipſi AB parallela. Tangentes ad puncta D et E ſemper concurrunt in recta AB, et tangentes ad puncta *d* et *e* in recta *ab*. Fig. 28.

§ 46. Contingat recta DT ſectionem in D, unde ducantur duæ quævis rectæ DE et DA, quæ ſectioni occurrant in E et A. Occurrat DE rectæ AK ſectionem contingenti, in K; et ductæ EN, KM tangenti DT parallelæ ſecent DA in N et M, ſumatur in rectæ DE, DR ad EN ut KM ad KE, & circulus ejuſdem curvaturæ cum ſectione in D tranſibit per R. Nam per Art. 15. eſt $\frac{QV^2}{DV^2 \times DR} = \frac{1}{DE} - \frac{1}{DK} = \frac{KE}{DE \times DK}$ et Fig. 29. n. 1.

$DR = \frac{DE \times DK}{KE} \times \frac{QV^2}{DV^2}$ (quoniam QV : DV ::
n. 2. KM : DK :: EN : DE) $= \frac{KM \times EN}{KE}$. Quod ſi fuerit tangens AK parallela rectæ DE, (i. e. ſi DE ſit ordinata diametri per A tranſeuntis) erit $DR = \frac{EN^2}{DE}$, vel DR ad DE ut EN^2 ad DE^2; ut alibi demonſtravimus Art. 373. tractatus de fluxionibus. Si in hoc caſu DE ſit diameter, erit $\frac{EN^2}{DE}$, adeoque DR, æqualis parametro diametri DE; ut ſatis notum eſt.

Fig. 30. n. 1. § 47. Ducantur rectæ DT, DE, quarum prior ſectionem conicam contingat in D, poſterior eidem occurrat in E. Ducatur DA quæ biſecet angulum EDT et ſectioni occurrat in A; jungatur AE, cui occurrat in V recta DV parallela rectæ quæ curvam contingit in A; et ductâ VR parallelâ rectæ DA, hæc ſecabit DE in R ubi circulus oſculatorius occurrit rectæ DE; eritque DR diameter curvaturæ ſi angulus EDT ſit rectus. Erit enim VR ad AD ut ER ad DE, et ut DR ad DK; unde DR ad DK ut DE ad EK, adeoque $\frac{1}{DR} = \frac{1}{DE} -$
n. 2. $\frac{1}{DK}$, ut oportebat, per Art. 15. Si autem ſit tangens AK parallela rectæ DE (quo in caſu tangentes AK et DT æquales conſtituunt angulos cum recta DA quæ proinde perpendicularis eſt axi figuræ) coincident puncta R et E, & circulus oſculatorius tranſibit per punctum E. Sequitur quoque ex dictis rectas EK, DE, et ER eſſe in progreſſione geometrica.

Fig. 31. § 48. Occurrat recta quævis DE ſectioni conicæ in D et E, concurrant rectæ curvam contingentes ad D et E in puncto V. Sit DOA diameter per D curvæ, & ſi conſtituatur angulus $DVr = EDO$, erit $DR\ (= 2Dr)$ chorda circuli oſculatorii. Ducatur enim AK ſectionem contingens quæ rectæ DE occurrat in K, et tangenti EV in Z; ducatur EN parallela tangenti DT rectam DA ſecans in N; cumque ſit DR ad KA ut EN ad EK; ſitque $KZ\ (= \frac{1}{2} AK)$ ad EK ut VD ad DE, erit VD ad DE ut $\frac{1}{2}DR$ ad EN; adeoque triangula DVr et EDN ſimilia et angulus DVr æqualis angulo EDO. Hanc methodum determinandi circulum oſculatorium demonſtravimus in tractatu de fluxionibus, Art. 375. ſed non adeo breviter.

§ 49. Variatio curvaturæ, ſive tangens anguli contactus ſectione conica & circulo oſculatorio comprehenſi, eſt directe ut tangens anguli contenti diametro quæ per contactum ducitur & normali ad curvam, & inverſe ut quadratum radii curvaturæ. Sit enim DR diameter curvaturæ, & hæc variatio ad punctum D erit ut $\frac{1}{DR \times DV}$, per Art. 17. adeoque, cum ſit DV ad Dr ut DE ad EN, ut $\frac{EN}{DE \times DR^2}$. Variatio autem radii curvaturæ eſt ut tangens anguli EDO. Quod ſi recta DO circulo oſculatorio occurrat in n, parabola diametro & parametro Dn deſcripta, quæque contingit rectam DT in D, ea erit cujus contactus cum ſectione eſt intimus, per Art. 19. Fig. 32.

§ 50. Cæteris manentibus, ex puncto V ducatur recta VH circulum oſculatorium contingens in H; jungatur HD, cumque ſit angulus RDH complementum anguli DrV ad rectum, erit $RDH = DVr = EDO$; adeoque variatio radii curvaturæ erit ut tangens anguli RDH; Fig. 32.

RDH; & coincidentibus rectis DR et DH variatio evaneſcit.

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SECTIO III.

De Lineis tertii Ordinis.

§ 51. DE lineis tertii ordinis ſive curvis ſecundi generis, uberius nobis agendum eſt. Doctrinam conicam, variis modis uſque ad faſtidium fere, tractarunt permulti. Hanc autem geometriæ univerſalis partem, pauci adtigerunt; eam tamen nec ſterilem eſſe nec injucundam ex ſequentibus, ut ſpero, patebit, cum præter proprietates harum figurarum a *Newtono* olim traditas, aliæ ſunt plures geometrarum attentione non indignæ. Oſtendimus ſupra, rectam ſecare poſſe lineam tertii ordinis in tribus punctis, quoniam æquationis cubicæ tres ſunt radices, quæ omnes reales eſſe poſſunt. Recta autem quæ lineam tertii ordinis in duobus punctis ſecat, eidem in tertio aliquo puncto neceſſario occurrit, vel parallela eſt aſymptoto curvæ, quo in caſu dicitur ei occurrere ad diſtantiam infinitam: æquationis enim cubicæ ſi duæ radices ſint realis, tertia neceſſario realis erit. Hinc recta quæ lineam tertii ordinis contingit, eam in aliquo puncto ſemper ſecat; cum contactus pro duabus interſectionibus coincidentibus habendus ſit. Recta autem quæ curvam in puncto flexus contrarii contingit, ſimul pro ſecante habenda eſt. Ubi duo arcus curvæ ſibi mutuo occurrent, punctum *duplex* formatur, & recta quæ alterum arcum ibi contingit in eodem puncto alterum ſecat. Recta autem alia quævis ex puncto duplice ducta in uno alio puncto curvam ſecat, ſed non in pluribus.

§ 52. Prop. I. Sint duæ parallelæ, quarum utraque ſecet lineam tertii ordinis in tribus punctis; recta quæ utramque parallelam ita ſecat ut ſumma duarum partium parallelæ ex uno ſecantis latere ad curvam terminatarum æqualis ſit tertiæ parti ejuſdem ex altero ſecantis latere ad curvam terminatæ ſimiliter ſecabit omnes rectas his parallelas quæ curvæ in tribus punctis occurrunt; per Art. 4.

§ 53. Prop. II. Occurrat recta poſitione data lineæ tertii ordinis in tribus punctis; ducantur duæ quævis parallelæ quarum utraque curvam ſecet in totidem punctis; & ſolida contenta ſub ſegmentis parallelarum ad curvam & rectam poſitione datam terminatis erunt in eadem ratione ac ſolida ſub ſegmentis hujus rectæ ad curvam & parallelas terminatis, per Art. 5.

Hæ duæ proprietates a *Newtono* olim expoſitæ fuerunt.

§ 54. Prop. III. Cæteris manentibus ut in propoſitione præcedente, occurrat recta poſitione data lineæ tertii ordinis in unico puncto A, & ſolidum ſub ſegmentis PM, Pm, Pμ unius parallelæ contentum erit ſemper ad ſolidum ſub ſegmentis pN, pn, $p\nu$, alterius parallelæ ut ſolidum AP × bP^2 contentum ſub ſegmento AP & quadrato Fig. 33.

drato diſtantiæ bP puncti P a puncto quodam b ad ſolidum $Ap \times bp^2$ contentum ſub ſegmento Ap et quadrato diſtantiæ puncti p ab eodem puncto b, per Art. 6.

Fig. 34. n. 1. § 55. PROP. IV. Ex dato quovis puncto P
ducatur recta PD quæ lineæ tertii ordinis occurrat in tribus punctis D, E, F, & alia quævis
recta PA quæ eandem ſecet in tribus punctis A,
B, C. Ducantur tangentes AK, BL, CM, quæ
rectæ PD occurrant in K, L, et M; et medium
harmonicum inter tres rectas PK, PL, PM, coincidet cum medio harmonico inter tres rectas
PD, PE, et PF, per Art. 10. & 28. Si autem
n. 2. recta PD curvæ occurrat in unico puncto D, inveniatur
punctum d ut in Art. 6. & medium
harmonicum inter tres rectas PK, PL, PM, erit
ad medium harmonicum inter duas rectas PD et
$\frac{1}{2}Pd$ in ratione 3 ad 2, per Art. 12.

§ 56. PROP. V. Revolvatur recta PD circa polum P, ſumatur ſemper PM in recta PD æqualis medio harmonico inter tres rectas PD, PE, et PF, eritque locus puncti M linea recta, per Art. 28.

Atque hæc eſt proprietas harum linearum a *Coteſio* inventa.

Fig. 35. § 57. PROP. VI. Sint tria puncta lineæ tertii ordinis in eadem recta linea; ducantur rectæ curvam in his punctis contingentes, quæ eandem ſecent in aliis tribus punctis; atque hæc tria puncta erunt etiam in recta linea.

Occurrat recta FGH lineæ tertii ordinis in tribus punctis F, G, et H. Rectæ FA, GB, HC, curvam in his punctis contingentes eandem ſecent in punctis A, B, C; & hæc puncta erunt in recta linea. Jungatur enim AB, & hæc tranſibit per C; ſi enim fieri poteſt, occurrat curvæ in alio puncto M, tangenti HC in N et rectæ FGH in P; cumque ſit $\frac{1}{PA} + \frac{1}{PB} + \frac{1}{PM} = \frac{1}{PA} + \frac{1}{PB} + \frac{1}{PN}$, per Prop. IV. erit $PN = PM$; quod fieri nequit niſi coincidant puncta N, M, et C. Recta igitur AB tranſit per C.

§ 58. *Corol.* Hinc ſi A, B, C, ſint tria puncta lineæ tertii ordinis in eadem recta linea, ductæ autem AF et BG curvam contingant in F et G, & juncta FG curvam denuo ſecet in H, juncta CH curvam continget in H. Si enim recta curvam contingeret in H quæ eandem ſecaret non in C ſed in alio quovis puncto, foret hoc punctum cum tribus aliis A, B, C, in eadem recta quæ igitur ſecaret lineam tertii ordinis in quatuor punctis. Hoc autem fieri non poteſt. Incidi autem primo in hanc propoſitionem via diverſa ſed minus expedita, eandem deducendo ex Prop. II. Similiter ſi recta A*f* curvam quoque contingat in *f*, & ducta G*f* curvæ occurrat in *h*, juncta C*h* erit tangens ad punctum *h*. Et ſi a punctis A, B, C, lineæ tertii ordinis in eadem recta ſitis, ducantur tot rectæ curvam contingentes quot duci poſſunt, erunt ſemper tres contactus in eadem rectâ.

§ 59. Prop. VII. Ex puncto quovis lineæ tertii ordinis ducantur duæ rectæ curvam contingentes, Fig. 36.

tingentes, & recta contactus conjungens denuo ſecet curvam in alio puncto, rectæ curvam in hoc puncto & in primo puncto contingentes ſe mutuo ſecabunt in puncto aliquo curvæ.

Ex puncto A ducantur rectæ curvam contingentes in F et G, juncta FG curvam ſecet in H, eandemque contingat in H recta HC quæ curvæ occurrat in C, & ducta AC erit curvæ tangens ad punctum A. Sequitur ex *Corollario* præcedente, coincidentibus enim A et B recta CA ſit tangens ad punctum A.

§ 60. *Corol.* 1. Si ex puncto curvæ C ducantur duæ rectæ eandem contingentes in A et H, & ex puncto alterutro A contingentes AF et AG ad curvam, recta per contactus F et G ducta tranſibit per alterum punctum H.

Fig. 37. § 61. *Corol.* 2. Contingat recta AC curvam in A, eamque ſecet in C, ductæ autem AF et CH curvam contingant in F et H, recta per contactus ducta eam denuo ſecet in G, & juncta AG curvam continget in G. Quod ſi alia recta C*b* ex puncto C ducatur ad curvam eam contingens in *b*; & junctæ *b*F, *b*G, curvæ occurrant in *f* et *g*, ductæ A*f* et A*g* erunt tangentes ad puncta *f* et *g*.

Fig. 38. § 62. *Corol.* 3. Sit A punctum flexus contrarii unde ductæ AF et AG curvam contingant in F et G juncta FG ſecet curvam in H, & ducta AH curvam continget in H. Si enim tangens ad punctum H curvæ in alio quovis puncto ab A diverſo occurreret, recta ex hoc occurſu ad punctum flexus contrarii A ducta curvam in A contingeret, quot fieri nequit. Manifeſtum autem eſt tres tantum duci poſſe rectas ex puncto flexus contrarii curvam contingentes præter eam quæ in hoc ipſo puncto ſimul tangit & ſecat, atque tres contactus cadere in eandem rectam lineam. Ex ſolo puncto flexus contrarii tres rectæ ductæ curvam ita contingunt ut tres contactus ſint in eadem rectâ. Sint enim F, G, H, in eadem recta, unde tangentes ductæ conveniant in eodem puncto curvæ *a*, quod non ſit punctum flexus contrarii; ducatur *ae* curvam contingens in *a*, quæque ei occurrat in *e*, & juncta *e*H curvam tanget in H, per hanc propoſitionem; adeoque rectæ *e*H et *a*H curvam contingerent in eodem puncto H. *q. e. a.*

§ 63. Prop. VIII. Ex puncto quovis lineæ tertii ordinis ducantur tres rectæ curvam contingentes in tribus punctis; recta duos quoſvis contactus conjungens occurrat denuo curvæ, & ex occurſu ad tertium contactum ducta curvam denuo ſecabit in puncto ubi recta ad primum punctum ducta curvam continget.

Ex puncto A, lineæ tertii ordinis ducantur tres rectæ AF, AG, et A*f*, curvam contingentes in tribus punctis F, G, et *f*; recta G*f* quæ horum duo quævis conjungit ſecet curvam denuo in N, et recta ex hoc puncto ad tertium contactum F ducta curvam ſecet in *g*, tum juncta A*g* curvam continget in *g*. Ducatur enim recta AC curvam contingens in A quæ eandem ſecet in C; cumque puncta G, N, et *f*, ſint in eadem recta, & tangentes ad puncta G et *f* tranſeant per A, ſequitur (per Prop. VII.) tangentem ad punctum N tranſire per C. Cumque puncta F, N, *g*, ſint in eadem rectâ, tangentes autem FA et NC curvæ occurrant in A et C, ſitque AC tangens ad punctum A, tangens ad punctum *g* tranſibit per A. Fig. 37.

§ 64. *Corol.* Hinc ſi curva deſcribatur, ex datis tribus punctis contactus ubi tres rectæ ex eodem puncto curvæ ductæ eam contingunt, invenitur quartum punctum contactus ubi recta ex eodem puncto curvæ ducta eam contingit. Atque hinc colligitur ex eodem curvæ puncto quatuor tantum rectas duci poſſe lineam tertii ordinis contingentes præter rectam quæ in hoc ipſo puncto curvam contingit. Si enim rectæ ex eodem curvæ puncto duci poſſent eam in quinque punctis contingentes, plures rectæ numero indefinitæ curvam contingentes ex eodem puncto duci poſſent; ut ex præmiſſis facile colligitur. Hoc autem Corollarium poſtea facilius demonſtrabitur. Vide infra, Art. 77.

Fig. 38. § 65. PROP. IX. Ex puncto flexus contrarii ducantur tres tangentes ad curvam, & recta contactus conjungens harmonice ſecabit rectam quamvis ex puncto flexus contrarii ductam & ad curvam terminatam.

Sit A punctum flexus contrarii, AF, AG, et AH, rectæ curvam contingentes in punctis F, G, et H. Ex puncto A ducatur recta quævis curvam ſecans in B et C, & rectam FH in P; eritque PB ad PC ut BA ad AC. Cum enim tres tangentes ad puncta F, G, et H, in eodem puncto A conveniant, erit per Prop. IV. $\frac{1}{BP} + \frac{1}{PA} - \frac{1}{PC} = \frac{3}{PA}$, adeoque $\frac{1}{PB} - \frac{1}{PC} = \frac{2}{PA}$, i. e. PA eſt medium harmonicum inter duas rectas PB et PC ad curvam terminatas. Quæ linearum tertii ordinis proprietas eſt ſimplicitatis inſignis.

§ 66. *Corol.* 1. Recta quæ duas quaſvis rectas ex puncto flexus contrarii ductas ad curvam ſecat harmonice, ſecabit quoque alias quaſvis rectas ex eodem puncto eductas & ad curvam terminatas.

§ 67. *Corol.* 2. Si recta aſymptoto parallela per punctum flexus contrarii ducta occurrat rectæ FH in R & curvæ in O, erit $\frac{1}{RO} = \frac{2}{RA}$, adeoque $RA = 2RO$.

§ 68. Prop. X. Recta duo puncta flexus contrarii conjungens vel tranſit per 3^{um} punctum flexus contrarii vel dirigitur in eandem plagam cum crure infinito curvæ Fig. 39.

Sint A et *a* puncta flexus contrarii, juncta A*a* curvæ occurrat in a, eritque a quoque punctum flexus contrarii. Si enim tangens figuræ in puncto a curvæ occurreret in alio quovis puncto *e*, forent A, *a*, *e*, in eadem rectâ. Verum ex hypotheſi ſunt A, *a*, et a in eadem recta, quæ igitur lineæ tertii ordinis occurreret in punctis quatuor. Sit A punctum flexus contrarii, & recta AO aſymptoto parallela curvæ occurrat in O ducatur OQ curvam contingens in O, & ſecans in Q, juncta AQ, tranſibit per D ubi curva aſymptoton ſecat.

§ 69. Prop. XI. Ductis ex puncto flexus contrarii A tangentibus ad curvam AF, AG, AH; & duabus ſecantibus quibuſcunque ABC, A*bc*, junctæ B*b* et C*c* vel B*c* et *b*C ſe mutuo ſecabunt in recta FH quæ contactus conjungit. Fig. 38.

Occurrat enim recta B*b* ipſi FH in Q, et BC eidem in P; jungantur QA et QC; cumque ſit AB ad AC ut PB ad PC, per Prop. IX. erunt QA, QB, QP et QC, harmonicales, adeoque A*b* ſecabit rectam QC in *c* et ipſam FH in *p*, ita ut A*b* ſit ad A*c* ut *pb* ad *pc*; & proinde erit *c* punctum curvæ, per Prop. IX. unde ſequitur converſe rectas B*b* et C*c* convenire in puncto Q rectæ FH; & ſimiliter oſtenditur rectas B*c* et *b*C ſibi mutuo occurrere in puncto *q* ejuſdem rectæ.

§ 70. *Corol.* 1. Ex puncto quovis Q rectæ FH ducantur ad curvam rectæ QB, QC, eam ſecantes in punctis B, *b*, M et C, *c*, N; tum junctæ CB, *cb*, MN, convenient in puncto flexus contrarii A; junctæ B*c* et *b*C, M*c* et N*b*, B*b* et C*c*, NB et MC, convenient in recta FH.

§ 71. *Corol.* 2. Tangentes ad puncta B et C conveniunt in puncto aliquo T rectæ FH; & ſi a puncto quovis T in recta FH ſito ducantur tangentes ad curvam, rectæ contactus conjungentes vel tranſibunt per punctum flexus contrarii, vel convenient in recta FH.

§ 72. *Corol.* 3. Dato puncto flexus contrarii A, & punctis B, C, *b*, *c*, ubi duæ rectæ ex eo ductæ curvam ſecant, datur recta FH poſitione; junctæ enim B*b* et C*c* occurſu ſuo dabunt punctum Q, & junctarum B*c* et *b*C occurſus dabit *q*, ductaque Q*q* ea eſt quæ contactus F, G, et H, conjungit. His autem quinque punctis datis cum aliis duobus M et *m*, determinatur linea tertii ordinis quæ per hæc ſeptem puncta A, B, C, *b*, *c*, M, *m*, tranſit & in puncto A habet flexum contrarium. Ex punctis enim M et *m* dantur N et *n*, ubi ductæ AM et A*m* curvam ſecant, & his novem conditionibus linea determinatur. Si autem dentur tria puncta M, *m*, et S; hæc dabunt tria alia N, *n*, et *s*; unde darentur undecim conditiones ad figuram determinandam, quæ nimiæ ſunt. Similiter dato puncto flexus contrarii A cum punctis F, G, (adeoque tangentibus AF et AG) et punctis M et *m* quibuſcunque, datur recta FG, adeoque puncta N et *n*, et determinatur curva.

§ 73. *Corol.* 4. Contingant rectæ HB, HC, curvam in B et C, et juncta CB tranſibit per A, junctæ CG et FB concurrent in puncto curvæ V, et ducta VH curvam continget in V. Tangens autem ad punctum flexus contrarii A determinatur ducendo AN cui occurrat in L, recta PL ipſi AH parallela & biſecanda PL in X; juncta enim AX erit tangens ad punctum A. Occurrat enim tangens ad A rectæ FH in S; eritque $\frac{1}{PS}+\frac{2}{PH}=\frac{1}{PH}+\frac{1}{PG}-\frac{1}{PF}$, adeoque $\frac{1}{PS}+\frac{1}{PH}=\frac{1}{PG}-\frac{1}{PF}$ (quoniam AC ſecantur harmonice in P et B, adeoque VA, VF, VP, et VG, harmonicales) $=\frac{2}{PK}$. Eſt igitur PK medium harmonicum inter PS et PH; unde ſi PL parallela rectæ AH occurrat rectis AV et AS in X et L, erit PX = XL. Fig. 40.

§ 74. Prop. XII. Ex puncto lineæ tertii ordinis A ducantur duæ rectæ curvam contingentes in F et G, juncta FG curvæ occurrat in H, & tangens ad punctum A ſecet curvam in M; jungatur HM, cui occurrat FLK ipſi AH parallela in L, & ſumatur FK = 2FL; tum juncta HK, recta quævis AB ex A ducta harmonice ſecabitur a rectis HK et HF in N, P, et a curva in B, C; ita ut NB erit ad NC ut BP and PC. Fig. 41.

Occurrat enim recta AB tangenti HM in T, eritque $\frac{1}{PB}+\frac{1}{PA}-\frac{1}{PC}=\frac{2}{PA}+\frac{1}{PT}$, adeoque $\frac{1}{PB}-\frac{1}{PC}$ $=\frac{1}{PA}+\frac{1}{PT}$ (per conſtructionem, & *harmonice*) $=\frac{2}{PN}$. Unde ſequitur rectam NC ſecari harmonice in punctis B et P, vel NB eſſe ad NC ut BP ad PC.

§ 75. *Corol.* 1. Hinc ſi duæ quævis rectæ ex A ductæ ſecentur in N harmonice ita ut PC ſit ad PB ut CN ad BN, omnes rectæ ex A eductæ a rectis HF et HK ſimiliter harmonice ſecabuntur.

§ 76. *Corol.* 2. Si curva punctum duplex non habeat, eamque ſecet recta HK in duobus punctis *f* et *g*, ductæ A*f* et A*g* erunt rectæ curvam contingentes in his punctis. Coincidat enim punctum B cum puncto N, quando N pervenit ad *f* occurſum rectæ HK cum curva; adeoque cum $\frac{1}{PB}\mp\frac{1}{PC}=\frac{1}{PN}$, erit $\frac{1}{PC}=\frac{1}{PN}$, et coincidit C cum B, et recta ex A ducta curvam tunc contingit. Ex altera parte, ſi recta A*f* curvam contingat tranſibit recta HK per *f*; ob æquales enim PB, PC, in hoc caſu, coincidunt puncta B et C cum N.

§ 77. *Corol.* 3. Si recta HK in ſolo puncto H curvæ occurrat, duæ tantum tangentes duci poterunt a puncto A ad curvam, viz. AF et AG. Quatuor tantum ad ſummum tangentes duci poſſunt a puncto quovis lineæ tertii ordinis ad curvam ut AF, AG, A*f*, A*g*. Si enim alia quavis tangens duci poſſet a puncto A ad curvam ut Aφ, recta HK tranſiret per punctum φ, et quatuor puncta lineæ tertii ordinis forent in eadem rectâ, viz. H, *f*, *g*, *φ*. *Q. E. A.*

§ 78. Prop. XIII. Si ex puncto lineæ tertii ordinis duci poſſunt quatuor rectæ curvam contingentes, rectæ contactus conjungentes convenient ſemper in puncto aliquo curvæ, & recta quævis a primo puncto ducta harmonice ſecabitur a curva & rectis binos contactus conjungentibus:

Sit A punctum curvæ, AF, AG, A*f*, et A*g*, rectæ curvam contingentes in punctis F, G, *f*, et *g*. Jungantur FG et *fg*, quibus occurrat recta quævis ABC (ex A ducta curvamque ſecans in B et C) in P et N; & recta NC harmonice ſecabitur in B et P, ita ut ſemper ſit NC ad NB ut CP ad PB: ſequitur ex Corol. 2. præcedentis. Rectæ autem FG et *fg* concurrunt in puncto curvæ H; & ſimiliter rectæ F*f* et G*g* conveniunt in E, atque F*g* et G*f* in R; et E, R erunt puncta curvæ, per idem corollarium. Atque hæc eſt poſterior duarum proprietatum linearum tertii ordinis quas deſcripſimus in tractatu de fluxionibus, Art. 402. Quod ſi recta AM curvam contingat in A, et ſecet in M, junctæ ME, MR, MH, curvam tangent in punctis E, R, H; & rectarum AE et HR, AR et HE, AH et RE occurſus erunt quoque in curva *.

§ 79. *Corol.* Cum igitur ſint rectæ HK, HB, HP, et HC, harmonicales; ſi rectæ HB et HC curvæ occurrant in *b* et *c*; erunt puncta A, *b*, et *c*, in eadem recta linea. Occurrat enim juncta A*b* curvæ in *b* et *c* atque ipſi HF in *p*, et HK in *n*; cumque ſit *nc* ad *nb* ut *pc* ad *pb* patet *c* eſſe in recta HC; & reciproce ſi *c* ſit in recta HC et *b* in recta HB, erunt A, *b*, *c*, in eadem rectâ.

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* Supple quæ deſunt in Schemate.

Fig. 42. § 80. PROP. XIV. Habeat linea tertii ordinis punctum duplex O. Ex puncto quovis curvæ A ducantur duæ rectæ AF et AG curvam contingentes in F et G; ducta FG curvam ſecet in H; jungatur OH. Recta quævis AB ex A ducta curvæ occurrat in punctis B et C, rectæ FG in P, & rectæ OH in N; & recta NP harmonice ſecabitur in punctis B et C, ita ut PB ſit ad PC ut BN ad NC.

Jungatur enim AO quæ rectæ FG occurrat in *p* et tangenti HL in *t*; cumque ſit O punctum duplex, erit $\frac{2}{pO}+\frac{1}{pA}=\frac{1}{pA}+\frac{1}{pt}$, adeoque $\frac{1}{pA}+\frac{1}{pt}=\frac{2}{pO}$. Secatur igitur *p*A harmonice in *t* et O, ita ut *pt* ſit ad *p*A ut *t*O ad OA, et harmonicales ſunt H*p*, H*t*, HO, et HA. Occurrat recta PA tangenti LH in T, cumque ſit $\frac{1}{PC}+\frac{1}{PB}+\frac{1}{PA}=\frac{2}{PA}+\frac{1}{PT}$, erit $\frac{1}{PC}+\frac{1}{PB}=\frac{1}{PA}+\frac{1}{PT}=\frac{2}{PN}$; conſequenter PC eſt ad NC ut PB ad BN.

§ 81. *Corol.* Si tangens HL occurrat rectæ GZ ipſi AH parallelæ in Z, & ſumatur GV=2GZ, ducta HV tranſibit per punctum duplex O, ſi modo curva tale punctum habeat. Vel ſi recta G*ra* occurrat rectis AH et HR in *a* et *r*, junctæ *r*A et R*a*, ſe decuſſent in *m*, juncta H*m* tranſibit per punctum duplex O.

§ 82.

§ 82. Prop. XV. Ex puncto lineæ tertii ordinis ducantur duæ tangentes, & ex alio quovis ejuſdem puncto ducantur rectæ ad contactus curvam in duobus aliis punctis ſecantes; tangentes ad hæc duo nova puncta in eodem puncto curvæ convenient.

Ex puncto A ducantur rectæ AF et AG curvam contingentes in F et G. Sumatur punctum quodvis curvæ P, jungantur PF et PG curvam ſecantes in punctis K et L; atque tangentes ad puncta K et L concurrent in puncto aliquo curvæ B. Determinatur autem punctum B, ducendo rectam PC quæ curvam contingit in P, et ſecat in C; ſi enim jungatur AC occurret denuo curvæ in puncto B. Fig. 43.

Cum enim puncta F, K, P, ſint in eadem rectâ, & tangentes ad puncta F et P curvam ſecent in A et C; ſequitur tangentem ad punctum K tranſituram per B. Et ob rectam LGP, tangens ad punctum L tranſibit quoque per B.

§ 83. *Corol.* Sint igitur A et B duo quævis puncta in linea tertii ordinis; ex utroque ducantur quatuor rectæ curvam in aliis quatuor punctis contingentes, viz. AF, AG, A*f*, A*g*; et BK, BL, B*k*, B*l*. Junctæ FK et GL, FL et GK, F*l* et G*k*, G*l* et F*k*; ſibi mutuo occurrent in quatuor punctis curvæ, P, Q, *q*, *p*; & ſi ducantur tangentes ad hæc quatuor puncta, hæ occurrent curvæ & ſibi mutuo in puncto C ubi recta AB curvam ſecat. Unde ſi ſint tria puncta lineæ tertii ordinis in eadem rectâ, & ex ſingulis ducantur quatuor rectæ curvam contingentes in quatuor aliis punctis, recta per duo quævis puncta contactus ducta curvam ſemper ſecabit in alio aliquo Fig. 44.

alio puncto contactus; & quatuor hujuſmodi rectæ per idem punctum contactus ſemper tranſibunt.

Fig. 43. § 84. PROP. XVI. Sint F et G puncta duo lineæ tertii ordinis, ita ſumpta ut rectæ FA et GA curvam in his punctis contingentes conveniant in puncto aliquo curvæ A. Sumatur in curva aliud quodvis punctum P, unde ducantur ad puncta F et G rectæ PF et PG quæ curvæ occurrunt in K et L; jungantur FL et GK, atque harum occurſus Q erit in curva. Tangentes autem ad puncta K et L ſibi mutuo & curvæ occurrent in puncto aliquo curvæ B, atque tangentes ad puncta P et Q convenient in puncto curvæ C, ita ut tria puncta A, B, C, ſint in eadem recta.

Ducatur enim tangens ad punctum P quæ curvæ occurrat in C, & ducta AC ſecet eandem in B; & ductæ BK, BL, erunt tangentes ad puncta K et L, per præcedentem. Occurrant recta LF curvæ in Q; & ſi recta GK non tranſeat per Q, occurrat curvæ in *q*. Quoniam igitur tria puncta L, F, Q, ſunt in eadem rectâ, tangentes vero ad L et F curvam ſecent in B et A, ſequitur (per Prop. VII.) tangentum ad punctum Q tranſire per punctum C. Similiter, cum ſint puncta G, K, et *q*, in eadem recta, tangentes autem ad puncta G & K tranſeant per A et B, tangens ad punctum *q* tranſibit quoque per punctum C. Utraque igitur recta CQ, C*q*, curvam contingit prior in Q, poſterior in *q*. Coincidunt igitur puncta Q et *q*, ſi enim diverſa eſſe ponamus, ſequitur per Prop. VIII. plures quam quatuor tangentes duci poſſe ad curvam ex eodem puncto C. Sint enim

A*ſ*

A*f* et A*g* rectæ quæ curvam contingant in *f* & *g*, & ductæ L*f*, L*g*, curvam ſecent in *m* & *n*; & rectæ C*m*, C*n*, erunt tangentes ad puncta *m* et *n*. Quare haberemus quinque tangentes ex C ad curvam ductas CP, CQ, C*m*, C*n*, & C*q*; quod repugnat Corol. 3. Prop. XII.

§ 85. *Corol.* 1. Dato puncto P, ubicunque ſumantur puncta F & G, modo tangentes ad hæc puncta in curva conveniant, datur punctum Q, ubi junctæ FL & GK occurrunt ſibi mutuo & curvæ. Et ſi a puncto P ducatur recta quævis PNM quæ curvæ occurrat in N et M, & junctæ QM, QN, eam ſecent in *m* & *n*; erunt puncta P, *n*, & *m*, in eadem recta linea. Oſtendimus enim tangentes ad puncta P & Q, ſe mutuo decuſſare in puncto curvæ *.

Fig. 43.

§ 86. *Corol.* 2. Si ſumantur quatuor puncta F, G, K, L, in linea tertii ordinis, ita ut tangentes ad puncta F & G, conveniant in aliquo puncto curvæ, & tangentes ad puncta K et L, conveniant quoque in aliquo puncto curvæ, ductæ FK & GL concurrent in puncto curvæ, & ductæ FL & GK ſibi mutuo occurrent in puncto curvæ.

§ 87. PROP. XVII. Sint F et G duo quævis puncta lineæ tertii ordinis, ubi ſi rectæ ducantur curvam contingentes, hæ ſe mutuo ſecabunt in puncto aliquo curvæ. Sumantur alia quatuor puncta curvæ L, K, *f*, *g*, ita ut ductæ LF et GK conveniant in curva, atque rectæ F*f* et G*g*, in eâ quoque conveniant; tunc ductæ L*f* et *g*K, ſe mutuo ſecabunt in curva, ut & ductæ L*g* et K*f*.

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* Supple quod deeſt in Schemate.

Tangentes enim ad puncta *f* et *g* ſe mutuo decuſſant in curva, per Prop. XIV. ut & tangentes ad puncta K et L, per eandem. Adeoque per Corol. 2. præcedentis, junctæ *f*L et K*g* conveniunt in curva, ut et *f*K et *g*L.

Fig. 45. n. 1. § 88. *Lemma.* Dentur tres rectæ IC, IH, et CH, poſitione; & tria puncta F, G, S, quæ ſint in eadem recta linea. Sumatur punctum quodvis Q in recta IC, juncta QF occurrat rectæ IH in L, & juncta QG rectæ HC in P; jungatur FP, ducta SL occurrat rectis FP et QP in *k* et N; atque puncta *k* et N erunt ad rectas poſitione datas. Jungatur enim IN, quæ occurrat rectæ GS in *m*, & ducatur per N parallela rectæ FS quæ occurrat rectis IC, IH, et LQ, in punctis *x*, *u*, et *r*; occurrat recta FG rectis IC, IH, et HC, in *a*, *b*, et *h*. Quoniam N*x* eſt ad N*r* ut G*a* ad GF, et N*r* ad N*u* ut SF ad S*b*, erit N*x* ad N*u* (adeoque *ma* ad *mb*) ut G*a* × SF ad GF × S*b*, i. e. in data ratione. Datur igitur punctum *m*, adeoque recta IN*m* poſitione; & ſimiliter eſt punctum *k* ad poſitione datam.

n. 2. § 89. *Corol.* Coincidentibus punctis S et G, coincidit quoque punctum *m* cum puncto G. Jungatur igitur IG quæ rectæ HC occurrat in D, & ducta CF occurrat rectæ HI in E, tum juncta DE erit locus puncti K ubi ductæ GL et FP ſe mutuo decuſſant.

Fig. 46. § 90. PROP. XVIII. Sit PGLFQK quadrilaterum inſcriptum figuræ, cujus ſex anguli tangant lineam tertii ordinis ut in Prop. XVI. Ducantur rectæ curvam contingentes IC, CH, HI, in tribus punctis Q, P, L, quæ non ſint in eadem rectâ; jungatur IG quæ tangenti CH occurrat

occurrat in D, et HF quæ tangenti CI occurrat in E; erunt puncta D, K, E, in eadem recta linea, quæ quidem curvam in puncto K contingit.

Supponamus enim rectas QFL et FKP moveri circa polum F, & rectas LGP et QKG circa polum G, puncta autem Q, L, et P, deferri in tangentibus QI, LI et PC; tum punctum K movebitur in recta DE, per Corol. præcedens. Unde ſi puncta Q, L, P, ferantur in curva quæ has rectas QI, LI, et PC, in his punctis contingit, movebitur quoque in curva quam recta DE contingit. Sed per Prop. XV. ſi puncta Q, L, P, ferantur in lineâ tertii ordinis propoſita, punctum K movebitur in eâdem, quam igitur recta DE contingit in K.

§ 91. *Corol.* 1. Similiter ſi rectæ AF et AG (quæ curvam contingunt in F et G) occurrant rectæ IH (quæ curvam contingit in L) in punctis M et N; juncta MP ſecet tangentem AG in *d*, & juncta QN tangentem AF in *e*; recta *de* tranſibit per K, & curvam in hoc puncto continget; atque quatuor puncta D, *d*, *e*, E, erunt in eadem rectâ linea.

§ 92. *Corol.* 2. Ex duobus punctis curvæ quibuſcunque C et B ducantur ad curvam quatuor contingentes binæ ex ſingulis, CQ et CP ex puncto C, BL et BK ex puncto B, ſintque harum tangentium occurſus I, H, E, et D; tum ductæ LQ et EH ſe mutuo ſecabunt in puncto curvæ F; atque junctarum LP et ID occurſus erit in puncto curvæ G; tangentes autem ad puncta F et G ſe mutuo ſecabunt in puncto curvæ A quod eſt in eadem recta cum punctis C et B.

§ 93. *Corol.* 3 Datis tribus punctis lineæ tertii ordinis quæ ſint in eadem rectâ, & duabus tangentibus ex horum ſingulis ductis ad curvam poſitione datis, ſex puncta contactus determinantur per hanc propoſitionem. Sint A, B, C, tria curvæ puncta data in eadem rectâ, AM et AN tangentes ex A, BMI, et BDE, tangentes ex B quæ prioribus occurrant in M, N, *e*, et *d*; ſintque CD et CE tangentes ex tertio puncto C ductæ; atque occurrat CD ipſis BM, BD, AM, et AN, in H, D, *h*, et *c*, & CE iiſdem in I, E, *n* et *m*. His poſitis, juncta N*e* ſecabit tangentem CI in puncto contactus Q, M*d* ſecabit tangentem CD in puncto contactus P, ID ſecabit tangentem AN in puncto contactus G, EH tangentem AM in contactu F, *mh* ſecabit tangentem BH in L, & denique *nc* tangentem BE in K. Quamvis autem problema in hoc caſu determinatum ſit, ſolutiones tamen plures admittit. Diverſæ enim lineæ tertii ordinis, ſed numero definitæ, per tria puncta A, B, et C, duci poſſunt contingentes ſex rectas poſitione datas AM, AN, BM, BD, CD, et CE. Occurrat enim N*e* tangenti CD in *p*, recta M*d* tangenti CE in *q*, ID tangenti AM in *f*, EH tangenti AN in *g*, *nc* tangenti BM in *l*, et *mh* tangenti BD in *k*; atque linea tertii ordinis quæ conditionibus propoſitis ſatisfacit continget rectas CD et CE vel in P et Q, vel in *p* et *q*. Ea continget rectas AM et AN vel in punctis F et G vel in *f* et *g*; rectas autem BM et BD vel in L et K, vel in *l* et *k*. Conſtat igitur plures lineas tertii ordinis problematis conditionibus ſatisfacere poſſe, ſed numero determinatas, adeoque problema eſſe determinatum *.

§ 94. *Corol.* 4. Datis duobus punctis lineæ tertii ordinis A et B, tangentibus quoque AM, AN, BM, BD poſitione datis cum tribus punctis contactus F, G, et L, datur punctum K ubi recta BD curvam contingit.

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* Supple quæ deſunt in Schemate.

Si enim ducantur rectæ N*e* et LF, harum occurſu dabitur punctum Q, & ducta QG ſecabit contingentem BD in puncto contactus K. Datur quoque punctum P occurſus rectarum LG et M*d*, vel rectarum M*d* et FK; tres enim rectæ LG, M*d*, et FK, neceſſario conveniunt in puncto P. Sit M*ed*N quadrilaterum quodvis, ſumatur punctum quodvis Q in diagonali N*e* et P in diagonali M*d*, recta quævis QFL ex Q ducta ſecet latera M*e* et MN in F et L, ducta PL ſecet latus N*d* in G, jungatur QG quæ latus *de* ſecet in K; atque puncta F, K, P, erunt ſemper in eadem recta linea, per ſuperius oſtenſa. Unde conſtat problema non ideo fieri impoſſibile, quod oporteat tres rectas LG, M*d*, et FK, in eodem puncto convenire.

§ 95. PROP. XIX. Sint D, E, F, puncta lineæ tertii ordinis in eadem recta, ſintque tres rectæ curvam in his punctis contingentes ſibi mutuo parallelæ. In recta DF ſumatur punctum P ita ut 2PF ſit medium harmonicum inter PD et PE; & ſi alia quævis recta per P ducta curvæ occurrat in *f*, *d*, et *e*, erit ſemper 2P*f* medium harmonicum inter P*d* et P*e*. Supponimus autem puncta *d* et *e* eſſe ad eaſdem partes puncti P, punctum autem *f* eſſe ad contrarias. Fig. 47.

Occurrant enim tangentes DK, EL, FM, rectæ *df* in punctis K, L, et M; eritque per Art. 9 $\frac{1}{Pf} - \frac{1}{Pd} - \frac{1}{Pe} = \frac{1}{PM} - \frac{1}{PK} - \frac{1}{PL}$ (ſi recta Q*q* tangentibus parallela harmonice ſecet rectam PD ita ut PE ſit ad EQ ut PD ad DQ, et Q*q* occurrat rectæ *fd* in $q) = \frac{1}{PM} - \frac{2}{Pq}$ (quoniam Pq eſt ad PM ut PQ ad PF, & ex hypotheſi $2PF = PQ$, adeoque $2PM = Pq$) $= \frac{1}{PM} - \frac{1}{PM} = 0$; unde $\frac{1}{Pf} = \frac{1}{Pd} + \frac{1}{Pe}$, adeoque $2Pf$ eſt medium harmonicum inter Pd et Pe.

§ 96. *Corol.* 1. Jungantur Dd et Ee quæ conveniant in puncto V, junctæ VQ et Ff erunt parallelæ; & productâ VQ donec occurrat rectæ fd in r, erit $Pf = \frac{1}{2}Pr$. Recta enim PD ſecatur harmonice in E et Q, ex hypotheſi, adeoque etiam recta Pd ſecatur harmonice in e et r, per Art. 21. unde $Pf = \frac{1}{2}Pr$; cumque ſit $PE = \frac{1}{2}PQ$; ſequitur rectam Ff parallelam eſſe harmonicali VQr.

§ 97. *Corol.* 2. Similiter ſi ſumatur in recta DF punctum p ita ut $2pD$ ſit æqualis medio harmonico inter pE et pF, & recta quævis ex p ducta curvæ occurrat in tribus punctis, erit ſegmentum hujus rectæ ex una parte puncti p ad curvam terminatum æquale dimidio medii harmonici inter duo ſegmenta eodem puncto p et curvâ ad alteras partes terminata.

Fig. 48. § 98. *Lemma.* Ex centro gravitatis trianguli ducatur recta quævis quæ tribus lateribus trianguli occurrat, & ſegmentum hujus rectæ centro gravitatis & uno trianguli latere terminatum erit dimidium medii harmonici inter ſegmenta ejuſdem rectæ centro gravitatis & duobus aliis trianguli lateribus terminata. Sit P centrum gravitatis trianguli VTZ, occurrat recta FDE per P ducta lateribus in F, D, E; ſintque puncta D et E ad eaſdem partes puncti P; eritque $\frac{1}{PF} = \frac{1}{PD} + \frac{1}{PE}$. Ducatur enim per punctum P, recta MPL lateri VZ parallela, quæ lateribus VT, ZT, occurrat in L et M et rectæ VN parallelæ lateri ZT in N; cumque ſit $MP=PL$, et $TL=2VL$, ob ſimilia triangula TLM, VLN, erit $LM = 2LN$, unde $LN = LP$, et $PN = 2PM$, proinde ſi PD occurrat rectæ VN in K, erit (per Art. 21. & 23.) $\frac{1}{PD} + \frac{1}{PE} = \frac{2}{PK} = \frac{1}{PF}$.

§ 99. Prop. XX. Contingant tres rectæ VT, VZ, TZ, lineam tertii ordinis tranſeatque eadem recta linea per tres contactus & per P centrum gravitatis trianguli VTZ; recta quævis per hoc centrum ducta curvæ occurrat in puncto c ex una parte & in punctis a et b ex altera ejuſdem centri gravitatis parte, eritque $2Pc$ medium harmonicum inter ſegmenta Pa et Pb.] Fig. 49.

Occurrat enim recta Pc lateribus trianguli VTZ in f, d, et e; & rectæ VN lateri TZ parallelæ in k; eritque $2Pf=Pk$, adeoque $\frac{1}{Pf} = \frac{2}{Pk} = \frac{1}{Pd} + \frac{1}{Pe} = \frac{1}{Pa} + \frac{1}{Pb} - \frac{1}{Pc} + \frac{1}{Pf}$, adeoque $\frac{1}{Pc} = \frac{1}{Pa} + \frac{1}{Pb}$, unde Pc eſt dimidium medii harmonici inter rectas Pa et Pb.

§ 100. Prop. XXI. Sit V punctum duplex in linea tertii ordinis, VT et VZ rectæ curvam in hoc puncto contingentes, quibus in T et Z occurrat recta TZ curvam contingens in F ita ut $FT=FZ$; jungatur FV, in qua ſumatur $FP=\frac{1}{3}FV$; & ſi recta quævis per P ductæ curvæ occurrat in tribus punctis a, b, c, quorum a et b ſint ad eaſdem partes puncti P, c ad partes con- Fig. 50.

trarias, erit ſemper $2Pc$ medium harmonicum inter ſegmenta Pa et Pb, ſeu $\frac{1}{Pc} = \frac{1}{Pa} + \frac{1}{Pb}$.

Cum enim biſecetur TZ in F, ſitque $FP = \frac{1}{3}FV$, manifeſtum eſt punctum P eſſe centrum gravitatis trianguli VTZ; cumque ſit punctum P in rectâ FV quæ per contactus tranſit, ſequitur propoſitio ex præcedente.

§ 101. *Corol.* 1. Si jungatur rectæ V*a*, V*b*, et F*c*, erit P quoque centrum gravitatis trianguli hiſce rectis contenti, ut et trianguli tribus rectis curvam in *a*, *b*, *c*, contingentibus comprehenſi; & ſi ductæ V*a* et V*b* occurrant rectæ F*c* in *m* et *n*, erit ſemper F*m* æqualis F*n*.

§ 102. *Corol.* 2. Recta per punctum duplex ducta parallela rectæ F*c* harmonice ſecabit ipſam P*a* in *k* ita ut P*a* erit ad *ak* ut P*b* ad P*k*; quæ vero ducitur a puncto *k* ad *x* occurſum rectarum curvam in *a* et *b* contingentium parallela eſt recta *cy* figuram contingenti in *c*.

§ 103. *Corol.* 3. Datis duobus punctis *a* et *c* ubi recta quævis ex P ducta curvæ occurrit, datur tertium *b*; jungantur enim V*a* et F*c* quæ ſibi mutuo occurrant in *m*; ſumatur F*n* ex altera parte puncti F æqualis ipſi F*m*, et juncta V*n* ſecabit rectam P*a* in *b*.

Fig. 51. § 104. PROP. XXII. Ducatur per punctum quodvis P recta quæ dirigatur in plagam crurum infinitorum & occurrat curvæ in punctis *a* et *c*; ducatur per idem punctum recta quævis curvam ſecans in punctis D, E, F, quæque rectis curvam in *a* et *c* contingentibus occurrat in *k* et *m*, atque aſymptoto cruris infiniti in l; & ſi puncta D, E, k, m, l, ſint ad eaſdem partes puncti P, punctum vero F ad contrarias, erit $\frac{1}{Pl} = \frac{1}{PD} + \frac{1}{PE} - \frac{1}{PF} - \frac{1}{Pk} - \frac{1}{Pm}$, ubi termini cujuſvis ſignum eſt mutandum quoties ſegmentum ad oppoſitas partes puncti P protenditur.

Sequitur ex Theor. I. Art. 9. eſt enim per hoc theorema $\frac{1}{Pk} + \frac{1}{Pl} + \frac{1}{Pm} = \frac{1}{PD} + \frac{1}{PE} - \frac{1}{PF}$.

§ 105. *Corol.* 1. Si recta PD ducatur per concurſum tangentium *ak* et *cm*; & ſumatur PM æqualis medio harmonico inter rectas PD, PE, PF, ſecundum Art. 28. erit $\frac{1}{Pl} = \frac{3}{PM} - \frac{2}{Pk}$, adeoque $\frac{2}{3}$PM erit medium harmonicum inter Pl et $\frac{1}{2}$Pk. Quod ſi tangentes *ak* et *cm* concurrant in ipſo puncto M, aſymptotos quoque per M tranſibit.

§ 106. *Corol.* 2. In caſu Prop. XIX. ubi tres contactus ſunt in eadem recta linea & tres tangentes parallelæ, ſumatur punctum P ut in Propoſitione XIX. ſitque *a*P*c* aſymptoto parallela, occurrant *ak* et *cm* tangentes rectæ PD in *k* et *m*, eritque $\frac{1}{Pl} = \frac{1}{Pk} + \frac{1}{Pm}$, ſive P$l$ æqualis dimidio medii harmonici inter Pk et Pm. Quod ſi tangentes *ak* et *cm* concurrant in eodem puncto rectæ PD, erit $Pl = \frac{1}{2}Pk$; quoniam vero in Prop. XIX. $\frac{1}{Pf} = \frac{1}{Pd} + \frac{1}{Pe}$, erit $Pa = Pc$. Fig. 47.

Fig. 49. § 107. *Corol.* 3. Idem dicendum eſt de caſu Prop. XX, ubi tres contactus D, E, F, ſunt in eadem rectâ quæ tranſit per P centrum gravitatis trianguli VTZ tangentibus contenti. Si autem altera rectarum curvam in *a* vel *c* contingentium (poſita *a*P*c* aſymptoto parallelâ) ſit rectæ DP parallela, abibit aſymptotos in infinitum, eritque crus parabolicum.

§ 108. *Corol.* 4. Iiſdem poſitis ac in Prop. XXI. Sit *c*P*a* aſymptoto parallela, occurrant tangentes *ak*, *cm*, rectæ VF in *k* et *m*, eritque $\frac{1}{Pl}=\frac{1}{Pk}+\frac{1}{Pm}$. Unde ſi curva diametrum habet, cum hæc neceſſario tranſeat Fig. 52. per punctum duplex V, per punctum curvæ F ubi biſecatur tangens TFZ, ſumatur ab F verſus V, $FP=\frac{1}{3}FV$, ducatur *c*P*a* aſymptoto parallela, & tangens *ak* quæ diametro occurrat in *k*, & ex altera parte puncti P ſumatur, ſuper rectam PV, $Pl=\frac{1}{2}Pk$, & recta per *l* ducta ordinatim applicatis parallela erit aſymptotos curvæ. Si vero tangens *ak* ſit diametro parallela, erit crus curvæ generis parabolici. Propoſitio *Newtoni* de ſegmentis rectæ cujuſvis tribus aſymptotis & curva terminatis facile ſequitur ex Art. 4. ut ab aliis olim oſtenſum eſt.

Fig. 53. § 109. PROP. XXIII. Ex puncto quovis D lineæ tertii ordinis ducantur duæ quævis rectæ DEI, DAB, quæ curvæ occurrant in punctis, E, I, et A, B; ducantur tangentes AK, BL, quæ rectæ DE occurrant in K et L. Sit DG medium harmonicum inter ſegmenta DE, DI, ad curvam terminata, atque DH medium harmonicum inter ſegmenta DK, DL, ejuſdem rectæ tangentibus abſciſſa. Sit DV medium geometricum inter

DG et DH, ducatur VQ parallela tangenti DT, quæ occurrat rectæ DA in Q; & ſi circulus ejuſdem curvaturæ cum linea tertii ordinis propoſitâ in puncto D occurrat rectæ DE in R, erunt HG, QV et 2DR continue proportionales.

Nam per Theor. II. (Art. 15.) eſt $\frac{QV^2}{DV^2 \times DR} = \frac{1}{DE} + \frac{1}{DI} - \frac{1}{DK} - \frac{1}{DL} = \frac{2}{DG} - \frac{2}{DH} = \frac{2DH - 2DG}{DG \times DH} = \frac{2HG}{DV^2}$ (quoniam $DV^2 = DG \times DH$;) unde $QV^2 = 2HG \times DR$, adeoque HG ad QV ut QV ad 2DR.

§ 110. *Corol.* 1. Sumatur igitur D*r* in recta DE tertia proportionalis rectis HG et ½QV, & perpendicularis rectæ DE ad punctum *r* ſecabit normalem tangenti DT ad punctum D in centro circuli oſculatorii ſive circuli ejuſdem curvaturæ cum linea propoſita, in puncto O. Si puncta E, I, K, L, ſint ad eaſdem partes ejuſdem puncti prout DH major eſt vel minor quam DG, i. e. prout medium harmonicum inter ſegmenta DK, DL tangentibus abſciſſa majus eſt vel minus medio harmonico inter ſegmenta DE, DI, ad curvam terminata.

§ 111. *Corol.* 2. Si angulus EDT biſecetur recta DA, erit $QV = DV$, et $2HG \times DR = DV^2 = DG \times DH$, adeoque HG ad DG ut DH ad 2DR.

§ 112. *Corol.* 3. Revolvatur recta DA circa polum D, manente rectâ DE, et HG, differentia mediorum harmonicorum DH et DG, augebitur vel minuetur in duplicata ratione rectæ VQ. Quippe ob datam chordam circuli oſculatorii DR, manet quantitas $\frac{QV^2}{HG}$ quæ æqualis eſt 2DR.

§ 113.

Fig. 54. § 113. *Corol.* 4. Si tangentium AK et BL altera, ut BL, ſit rectæ DE parallela, ducantur GX et KZ parallelæ rectæ DT curvam in D contingenti, quæ ipſi AB occurrant in X, Z; eritque $\frac{GX \times KZ}{DG \times DK \times DR} = \frac{1}{DE} + \frac{1}{DI} - \frac{1}{DK} = \frac{2}{DG} - \frac{1}{DK} = \frac{2DK - DG}{DG \times DK}$, adeoque $\frac{GX \times KZ}{DR} = 2DK - DG$, & proinde erit ut $2DK - DG$ ad KZ ita GX ad DR. Si tangens AK evadat quoque parallela rectæ DE (quod in his figuris contingere poteſt) erit DG ad GX ut GX ad 2DR; nam in hoc caſu $\frac{GX^2}{DG^2 \times DR} = \frac{2}{DG}$, adeoque $GX^2 = DG \times 2DR$.

§ 114. *Corol.* 5. Si recta DE ſit aſymptoto parallela, adeoque curvæ occurrat in uno puncto E præter ipſum D, ſitque ſimul tangens BL aſymptoto parallela, ducatur EY parallela tangenti DT quæ occurrat rectæ DA in Y, eritque KE ad KZ ut EY ad DR *.

§ 115. *Corol.* 6. Si ſit D punctum flexus contrarii, coincidet punctum H cum G, evaneſcente linea HG, adeoque evadit DR infinite magna, i. e. curvatura minor eſt ad punctum flexus contrarii quam in circulo quantumvis magno; ut alibi quoque oſtendimus, tractatus de fluxionibus, Art. 378.

Fig. 55. § 116. *Corol.* 7. Sit V punctum duplex, DA aſymptoto parallela, & occurrant rectæ VQ, KZ, tangenti DT parallelæ rectæ DA in Q et Z, atque occurrat DV aſymptoto in L, ſitque DH medium harmonicum inter DK et DL, eritque $2DH - DG$ ad KZ ut DL ad DK,

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* Supple figuram.

atque VH : HN :: VQ : DR. Si recta DA biſecet angulum TDV, erit DR : DV :: DH : 2VH.

Fig. 56.

§ 117. Prop. XXIV. Sit D punctum quodvis lineæ tertii ordinis, occurrat tangens ad D curvæ in I, ſitque DS diameter circuli oſculatorii, quæ curvæ occurrat in A et B; unde rectæ ductæ curvam contingentes ſecent DI in K et L; ſit DH medium harmonicum inter DK et DL, & ſumatur DV ad DI ut DH ad differentiam rectarum 2DI et DH; eritque variatio curvaturæ inverſe et rectangulum SD×DV; & junctâ VS, variatio radii curvaturæ ut tangens anguli DVS.

Nam per Theor. III. (Art. 17.) variatio curvaturæ eſt ut $\frac{1}{DS} \times \overline{\frac{1}{DK} + \frac{1}{DL} - \frac{1}{DI}} = \frac{1}{DS} \times \overline{\frac{2}{DH} - \frac{1}{DI}} = \frac{1}{DS} \times \frac{2DI - DH}{DH \times DI} = \frac{1}{DS \times DV}$. Variatio autem radii oſculatorii eſt ut $\frac{DS}{DV}$, adeoque ut tangens anguli DVS, per Art. 18. parabola autem quæ eandem habebit curvaturam & eandem variationem curvaturæ cum linea propoſita, determinatur ut in Art. 19.

Fig. 57.

§ 118. *Corol.* Si tangens BL ſit tangenti ad D parallela, erit DV ad DI ut DK ad IK; & ſi utraque tangentium AK, BL, fiat parallela ipſi DT, erit DV =DI, adeoque variatio curvaturæ inverſe ut DS × DI. Quod ſi in hoc caſu ſit DT parallela aſymptoto curvæ, Fig. 58. evaneſcet variatio curvaturæ. Quemadmodum igitur evaneſcet variatio curvaturæ in verticibus axium ſectionum conicarum; ea ſimiliter evaneſcit in verticibus diametrorum linearum tertii ordinis quæ ad rectos angulos ordinatim applicatas biſecant.

Fig. 59. *Schol.* Sunt autem alia plurima theoremata de tangentibus & curvatura linearum tertii ordinis. Sint, *ex. gr.* F et G duo puncta lineæ tertii ordinis unde tangentes ductæ concurrunt in curva in A. Producatur FG donec curvæ occurrat in H. Sit TAC tangens ad punctum A, & conſtituatur angulus $FAN = GAT$ ad contrarias partes rectarum FA, GA, ſecetque AN rectam FG in N. Et ſi circuli oſculatorii occurrunt rectæ FG in B et *b*, erit GB ad F*b* ut rectangulum NFH ad NGH. Sit enim puncta *a* ipſi A quamproximum, & puncta *f*, *g*, *h*, ipſis F, G, H, quamproxima, eritque $AFa : FGf :: GF : FB$. $FGf\,(=HGh) : HFb :: FH : GH$. $HFb\,(=GFg) : AGa :: bG : GF$; unde $AFa : AGa :: FH \times bG : FB \times GH :: GN : FN$; unde $FB : Gb :: NFH : NGH$. *Sed de his ſatis.*

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Appen^{x}. pa. 2.

Fig. 1.

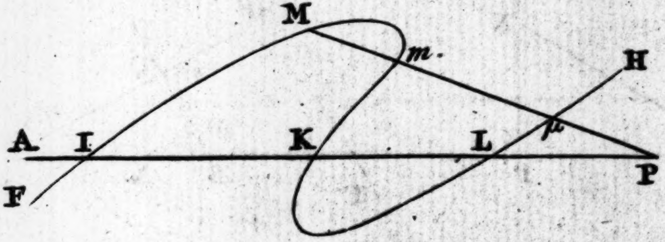


Fig. 4.

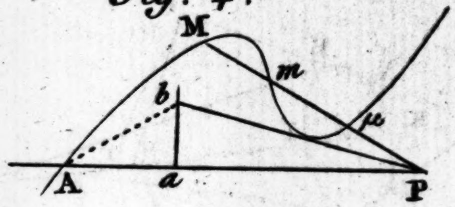
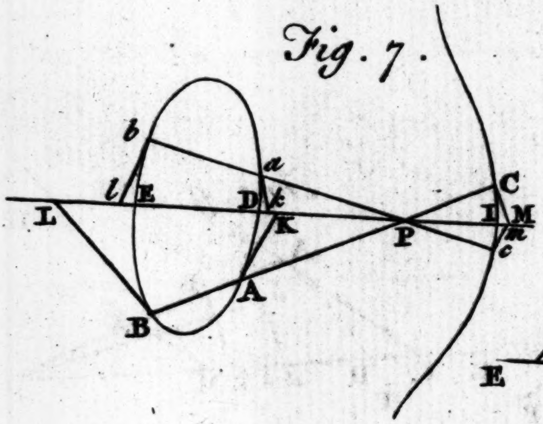


Fig. 7.



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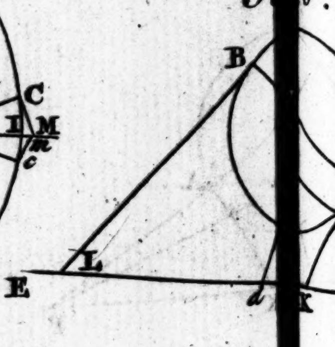

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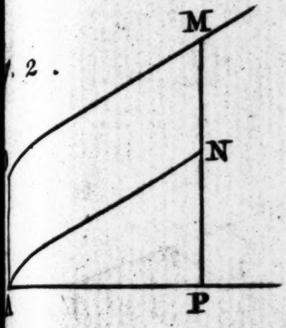

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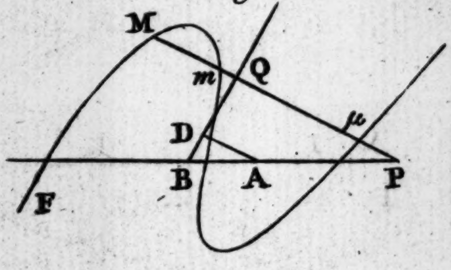

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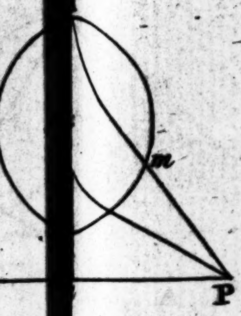

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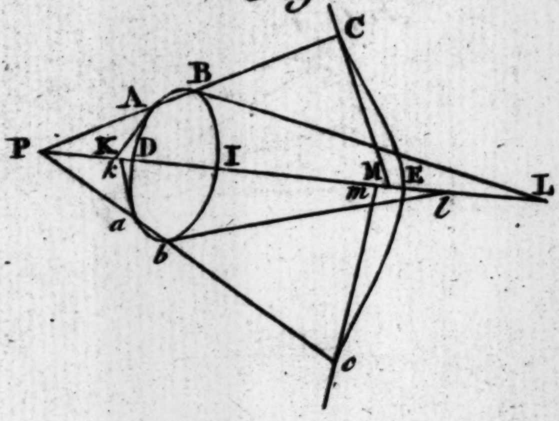

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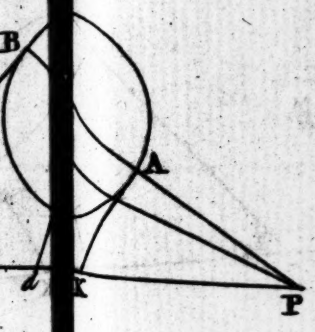

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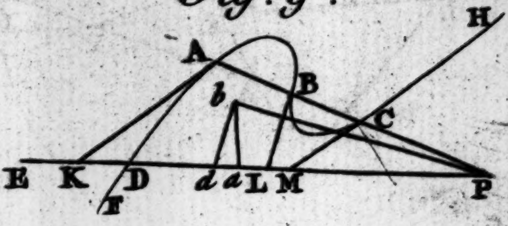

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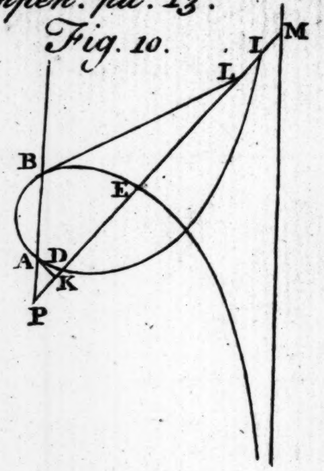
Fig. 10.



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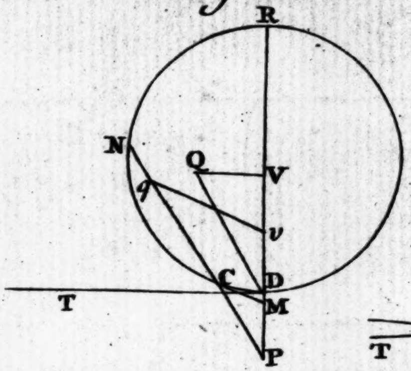

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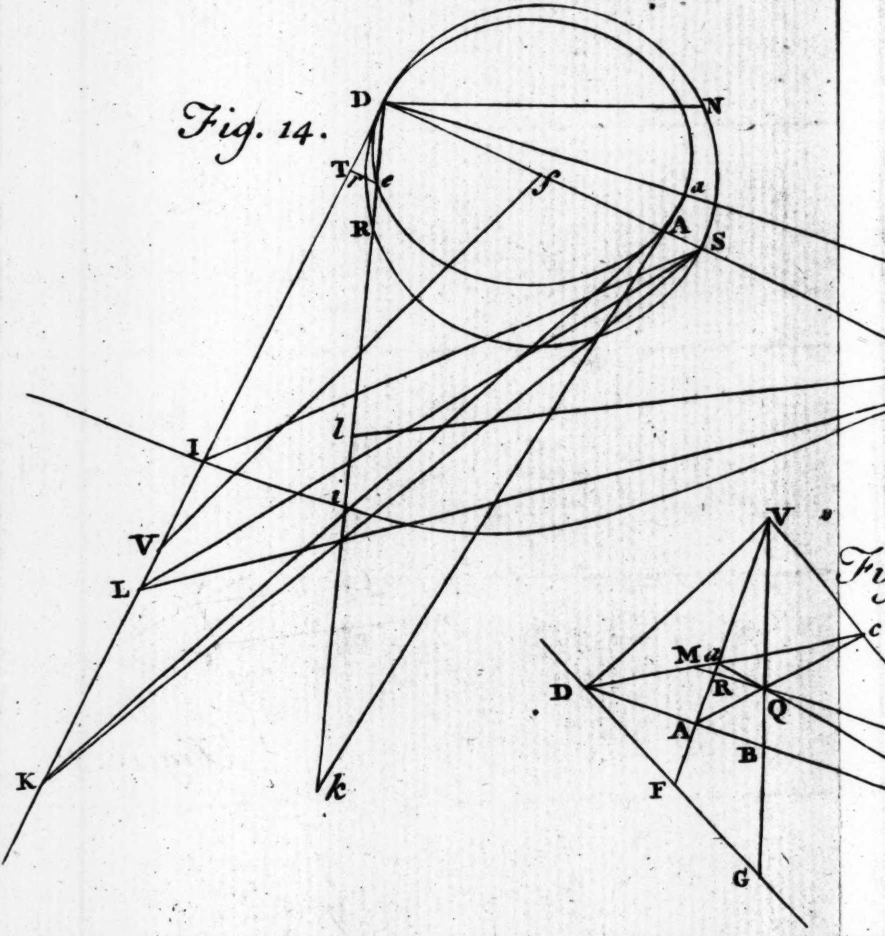

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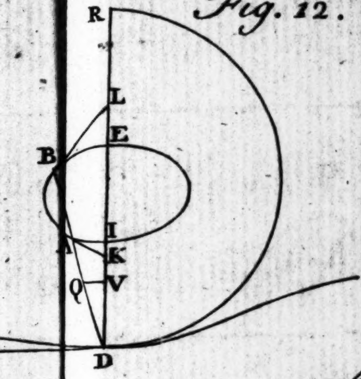
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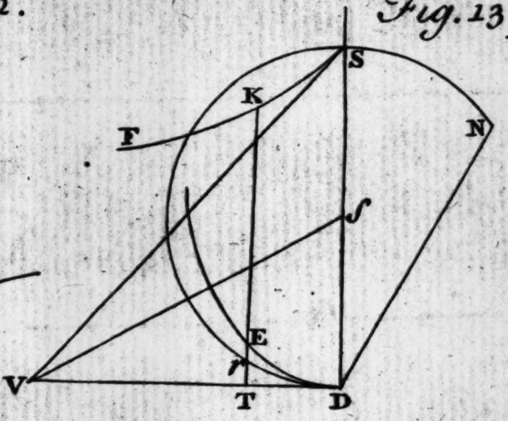
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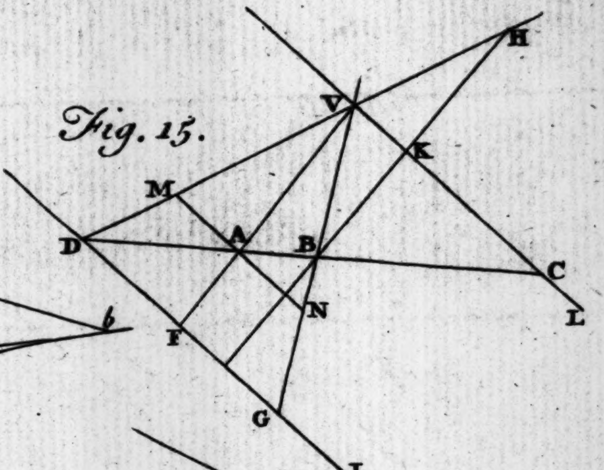
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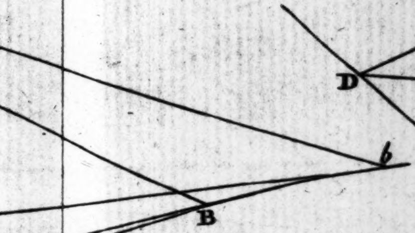

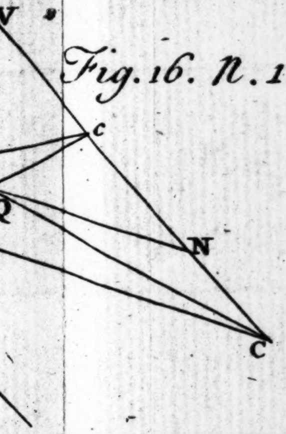
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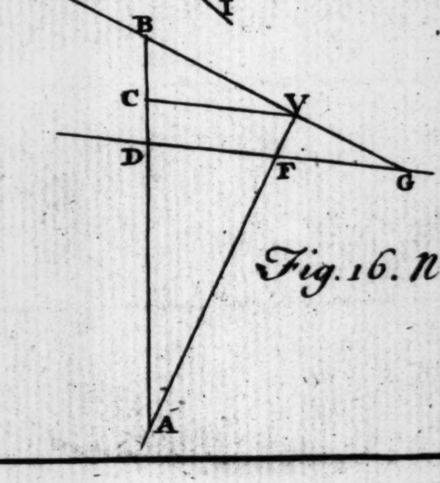
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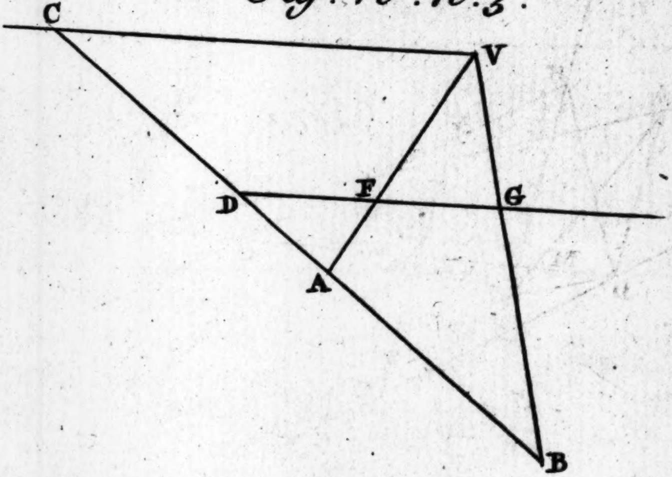

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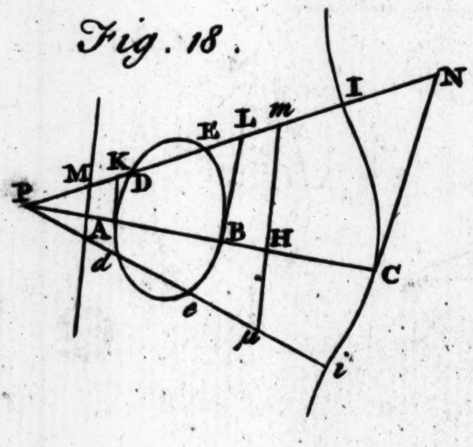

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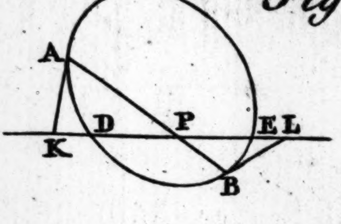

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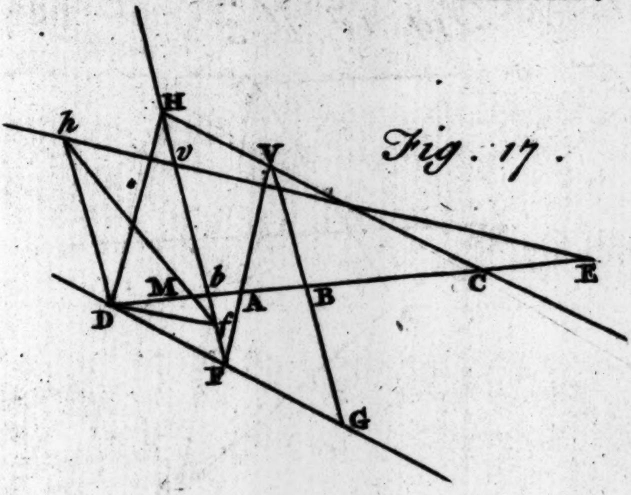
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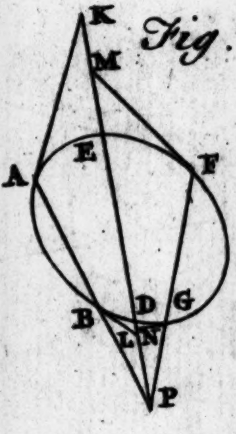
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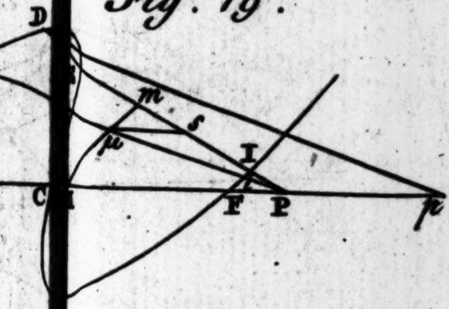
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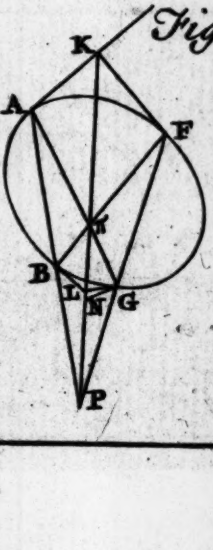
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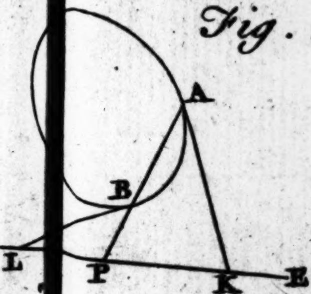
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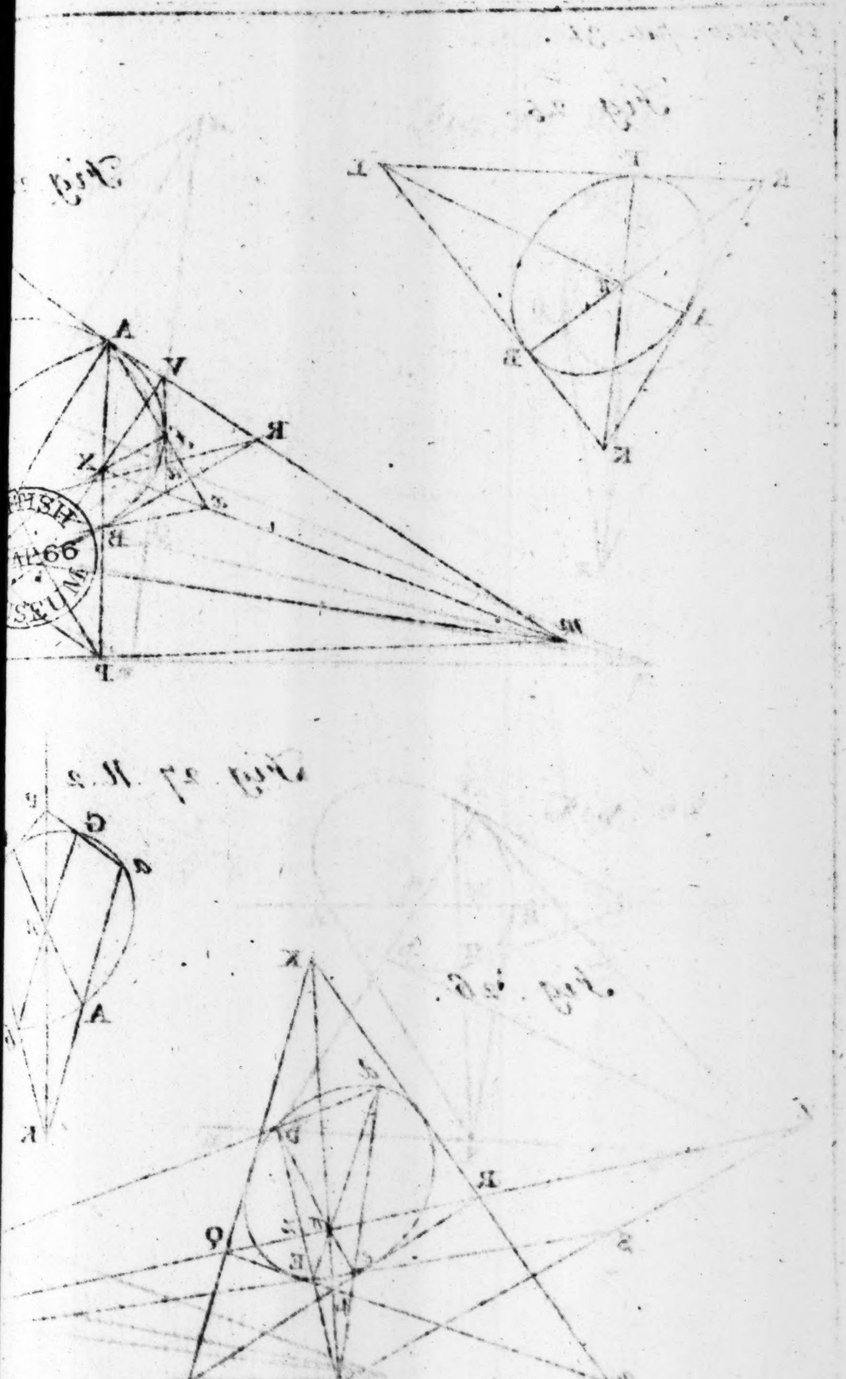

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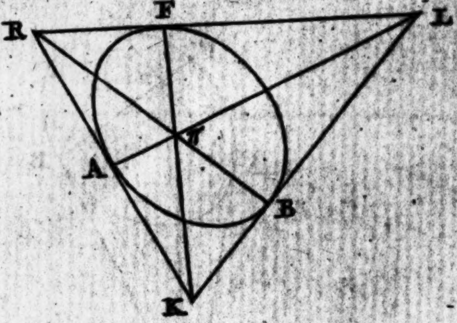
Fig. 25.



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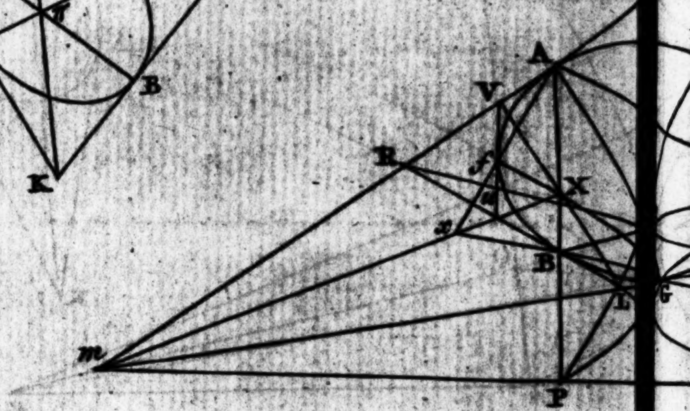

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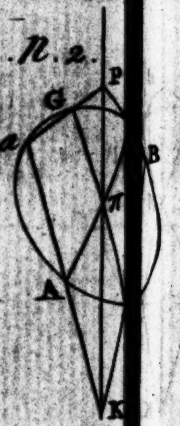

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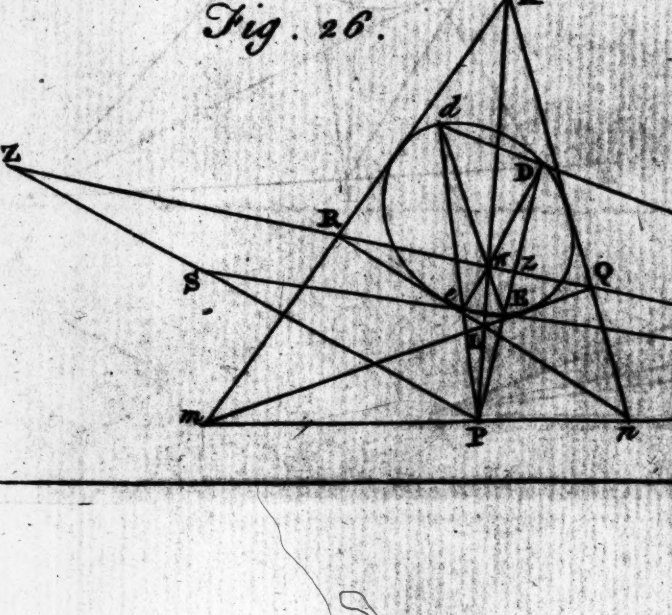

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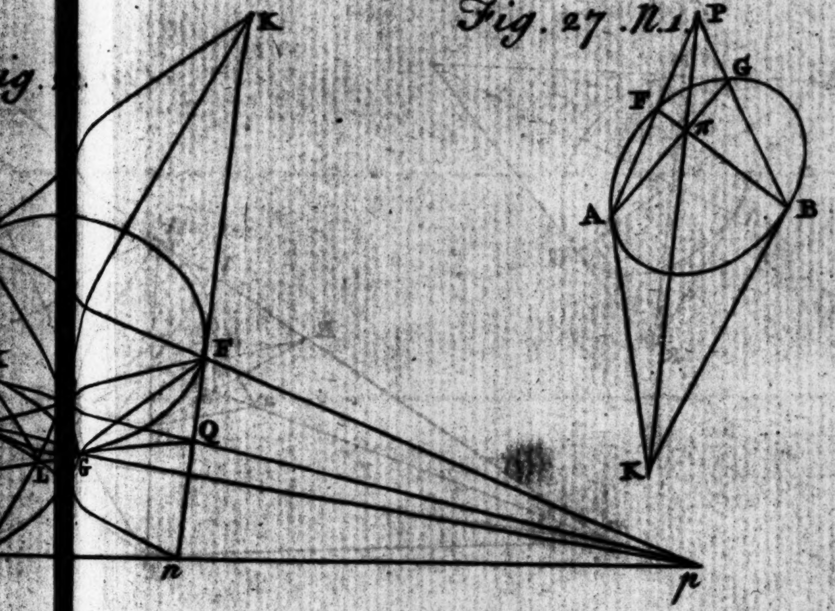

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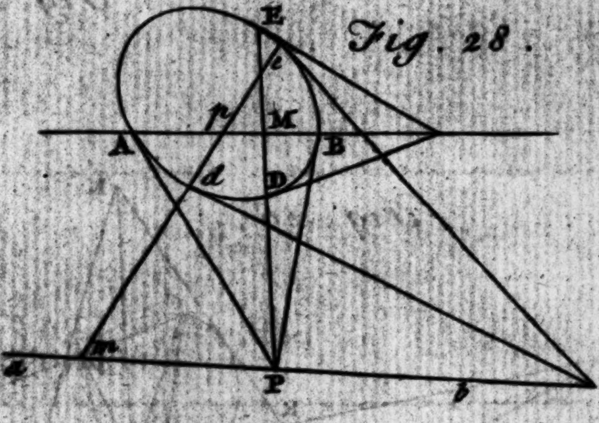

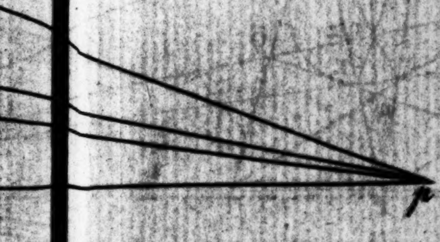

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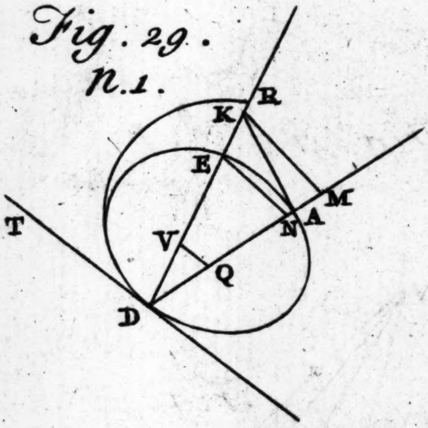

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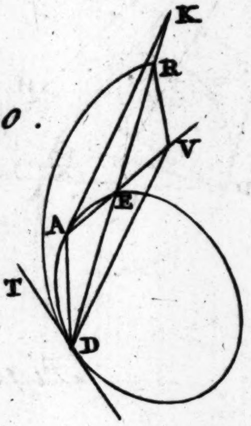

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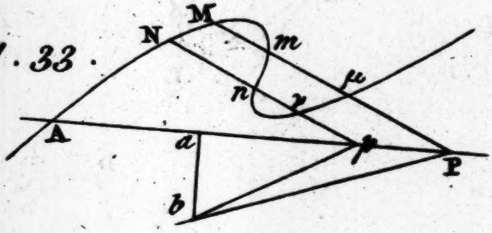

Fig. 29.
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Fig. 30.
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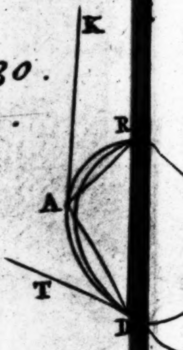

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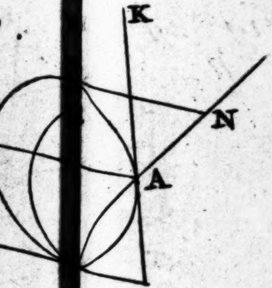

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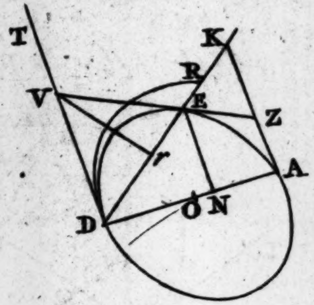

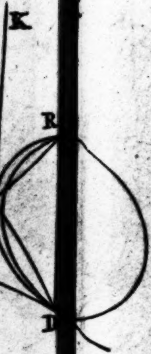

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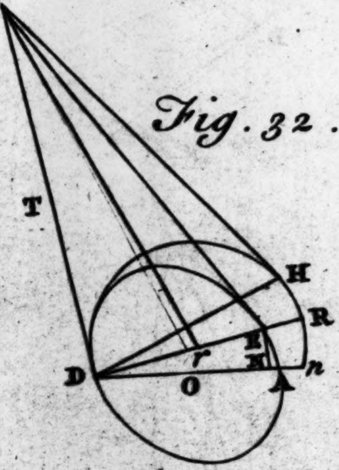

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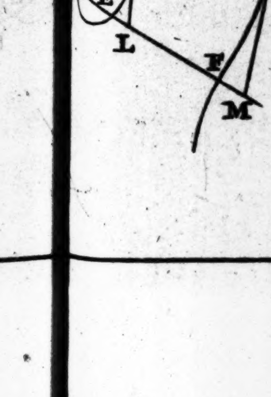

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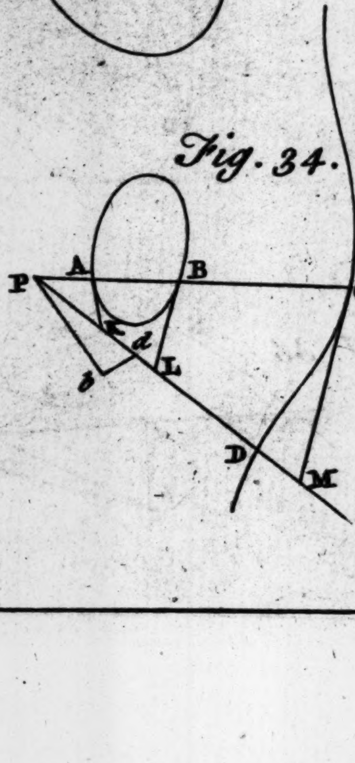

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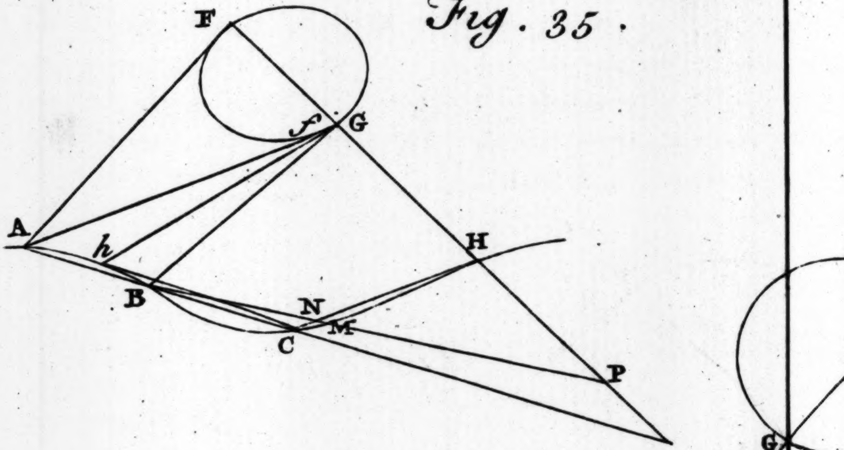

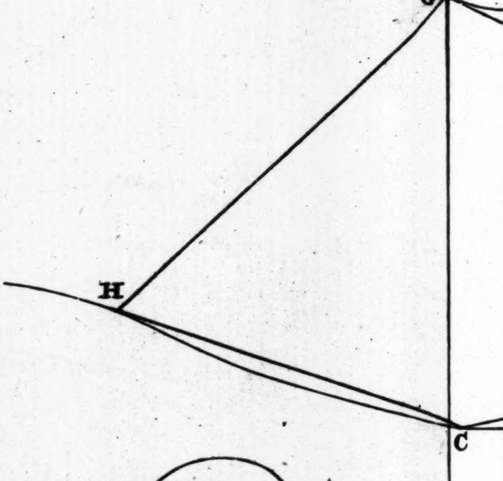

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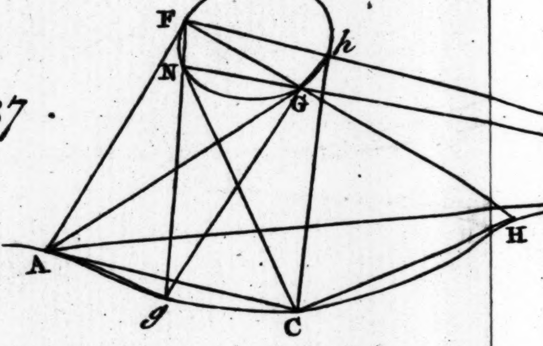

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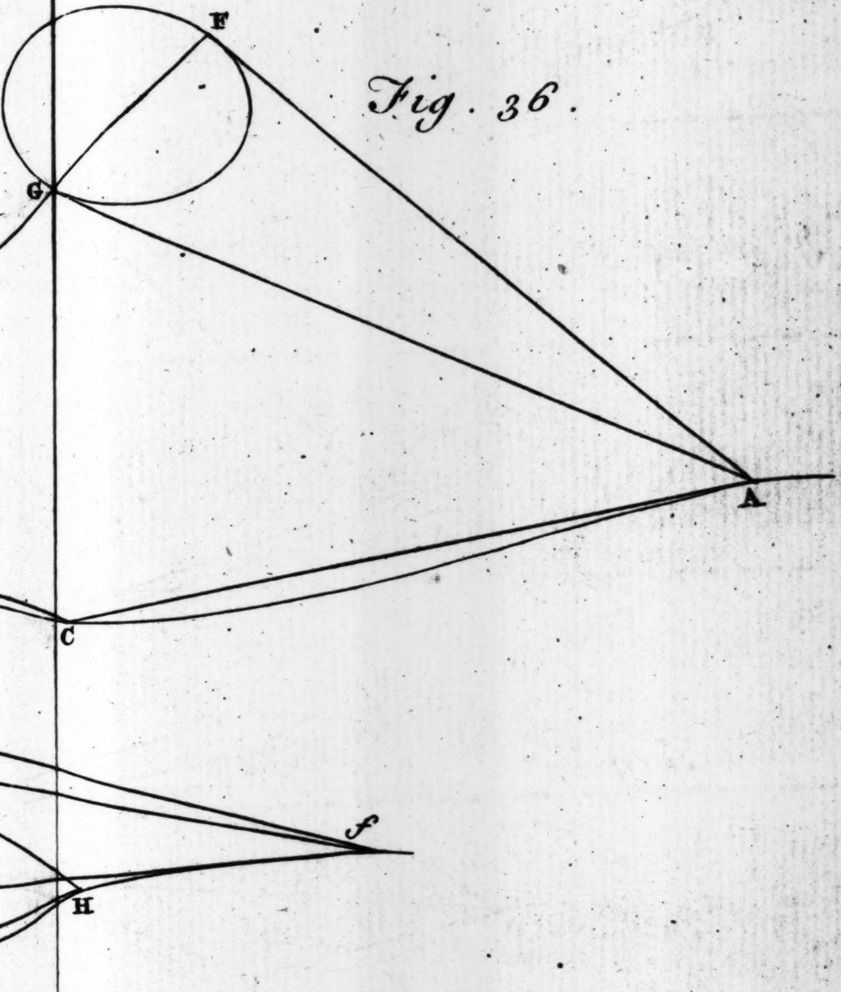

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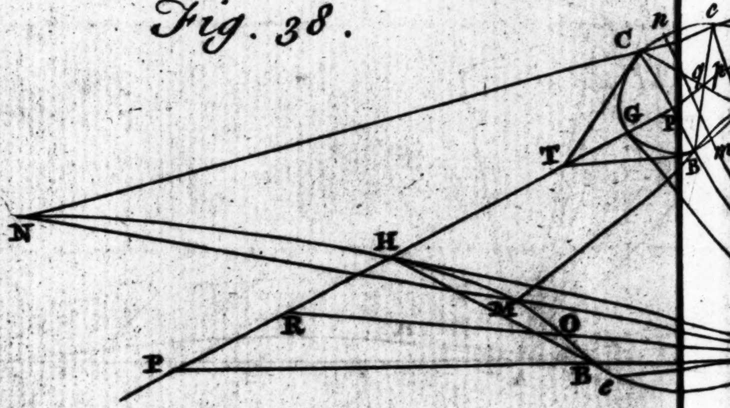
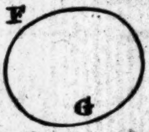
Fig. 38.





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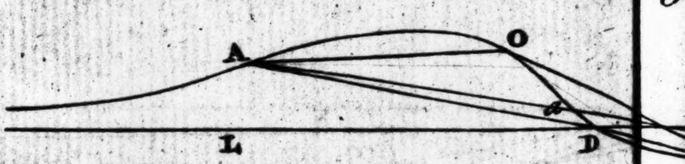

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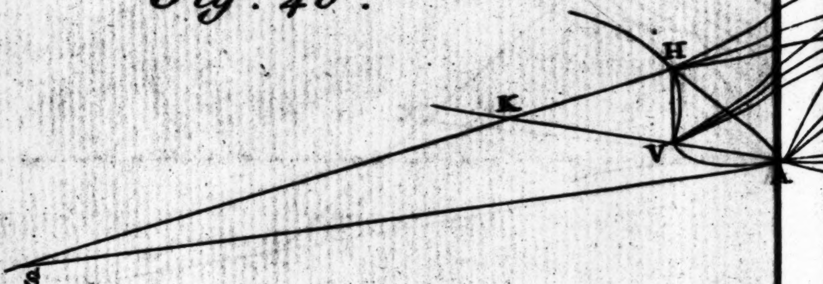

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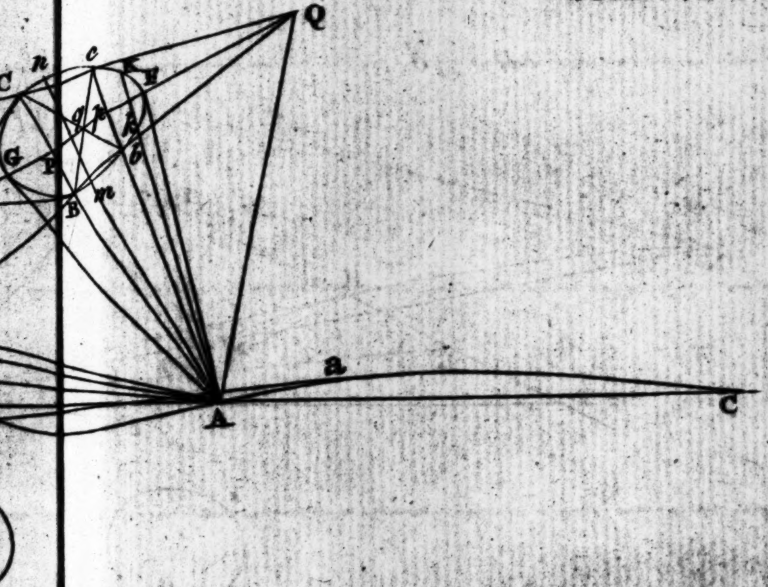

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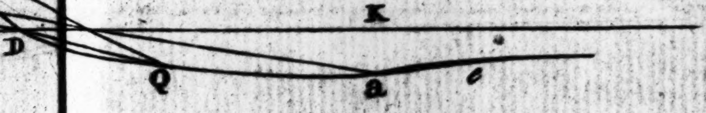

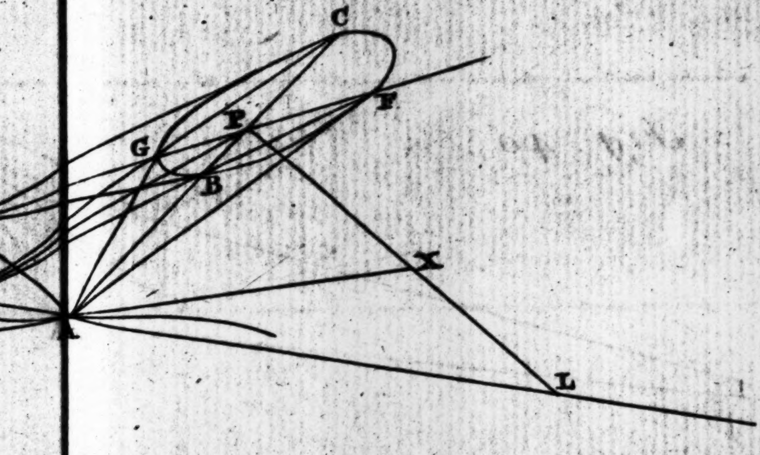

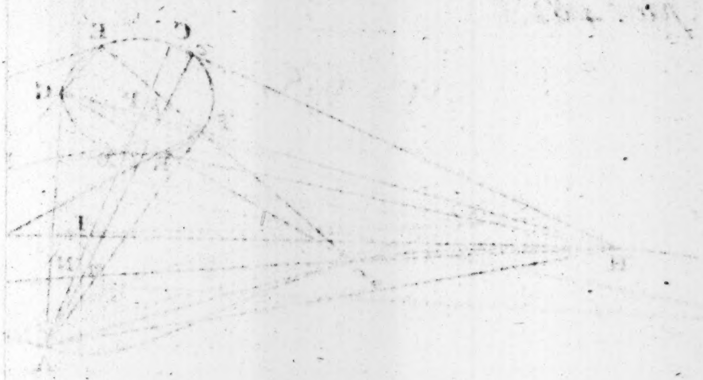

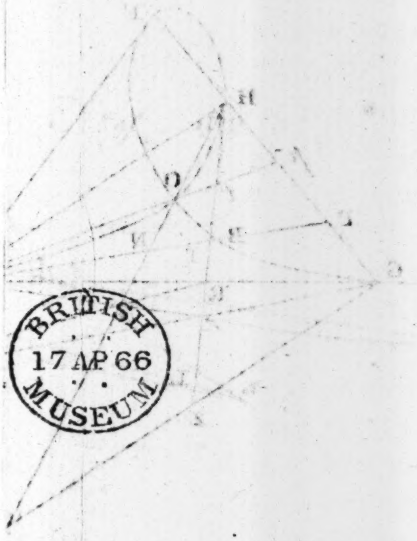

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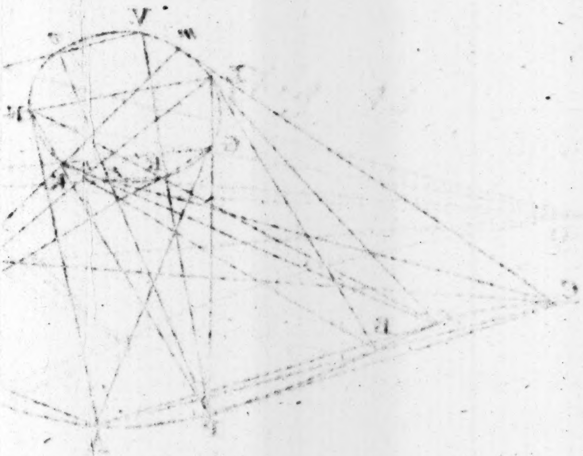

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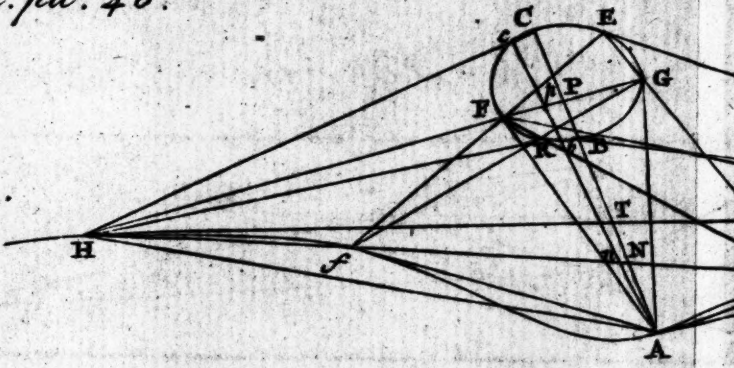

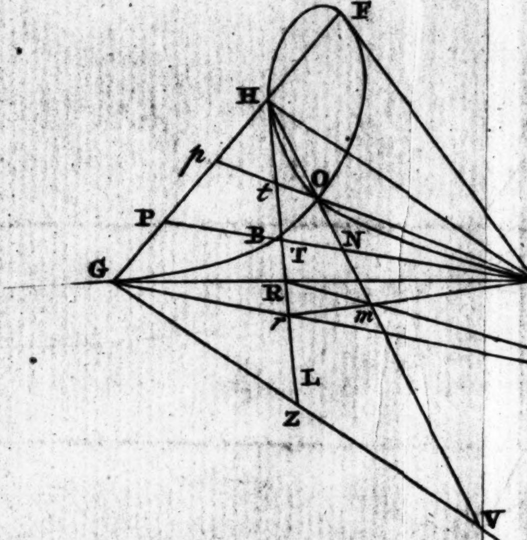

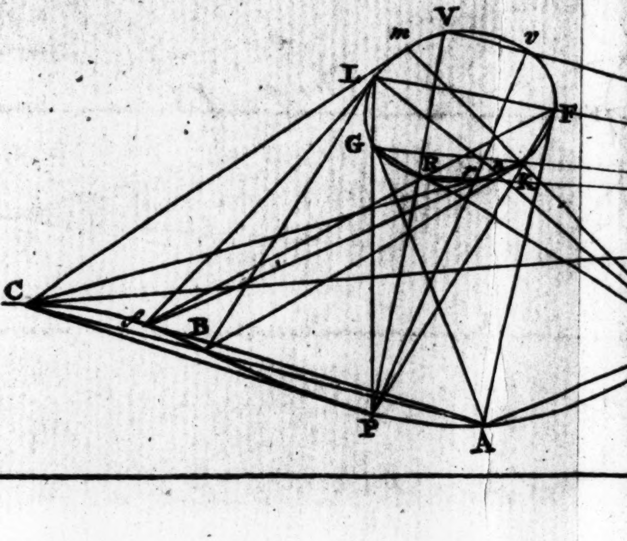

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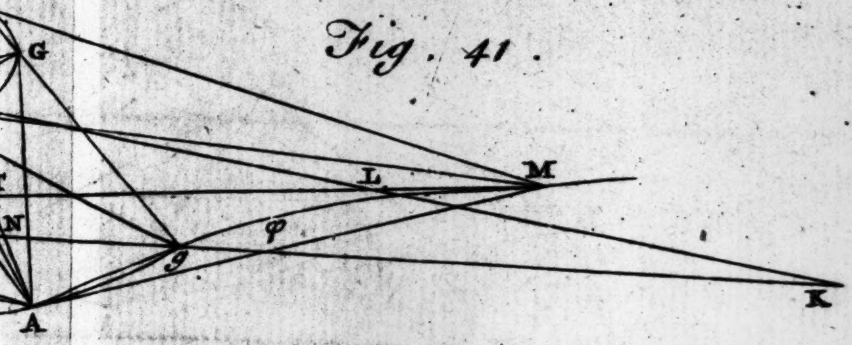

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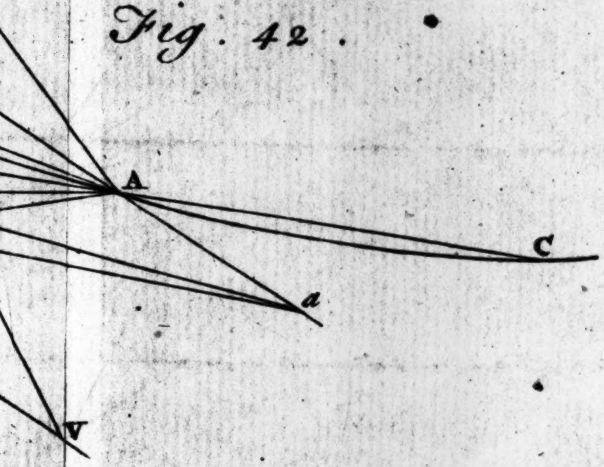

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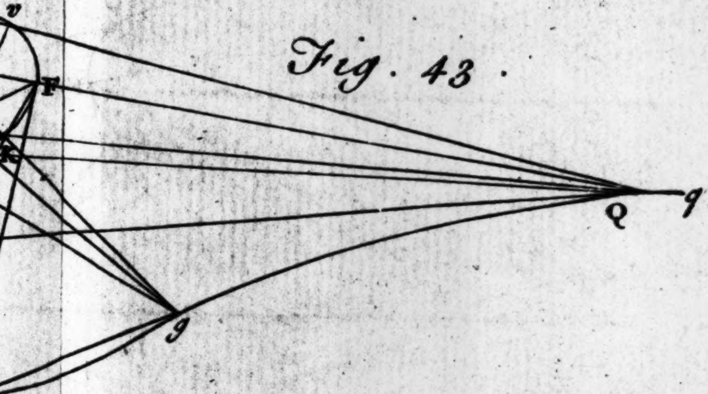

Fig. 43.

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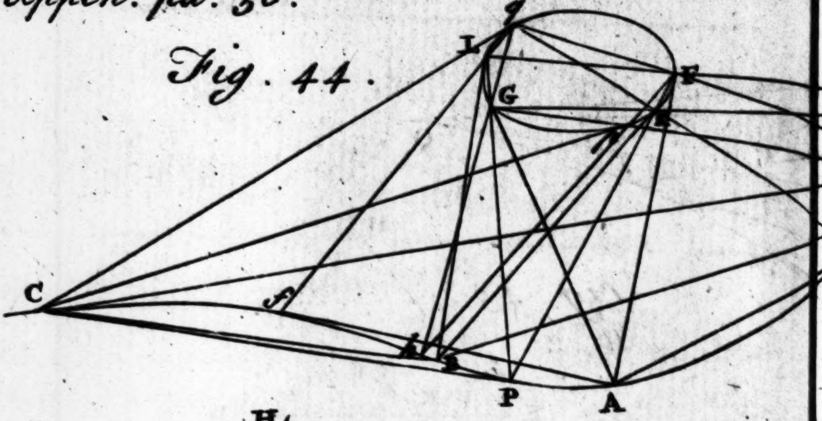
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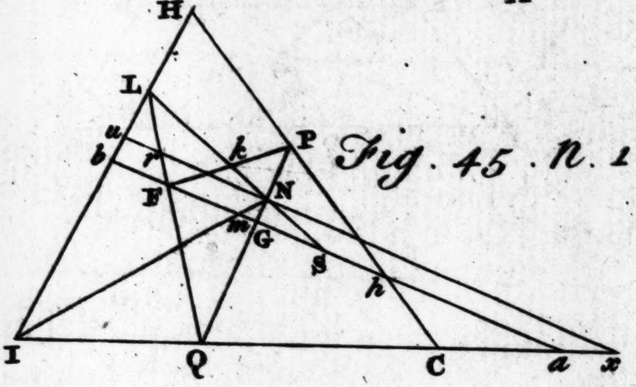
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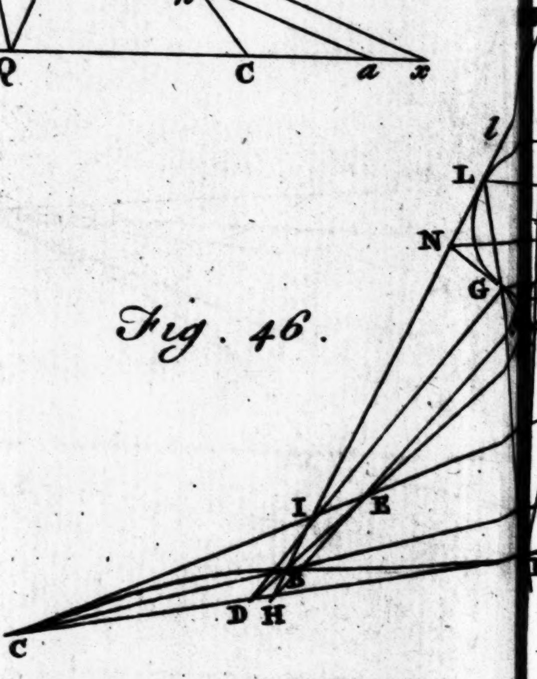
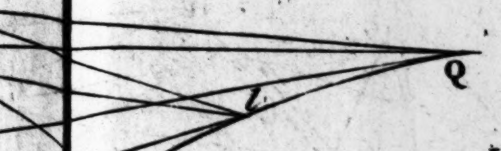


Fig. 46.

TAB. IX.



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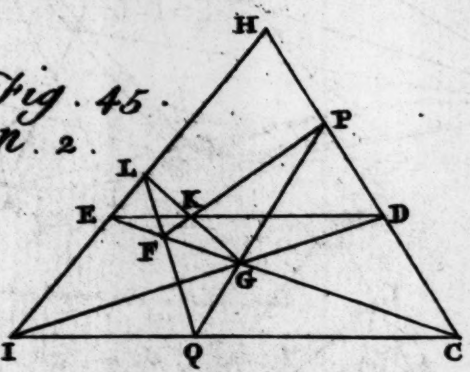

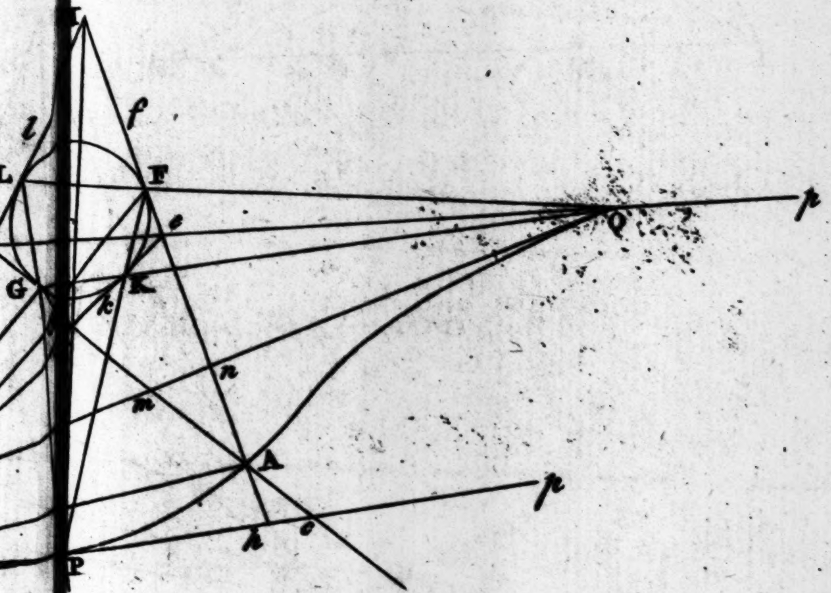

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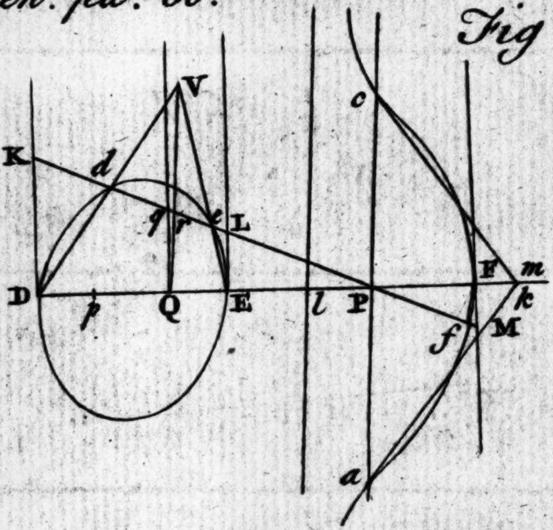
Fig. 47.



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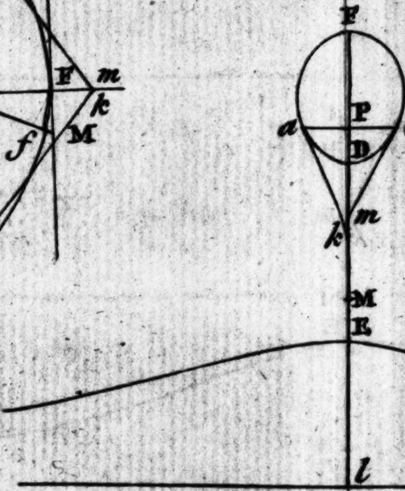

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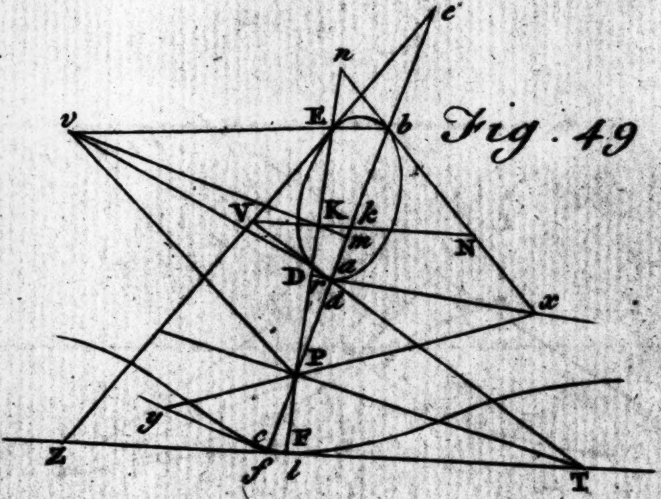
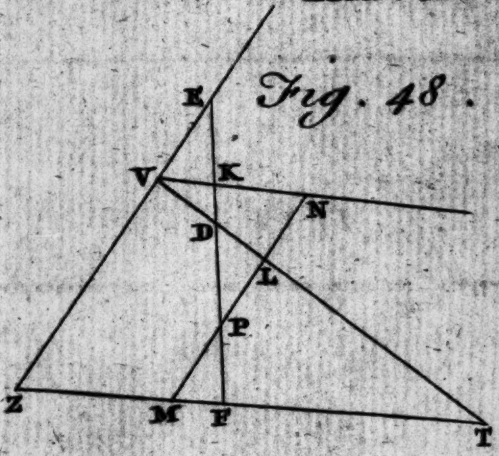


TAB. X.

Fig. 48.



ig. 51.

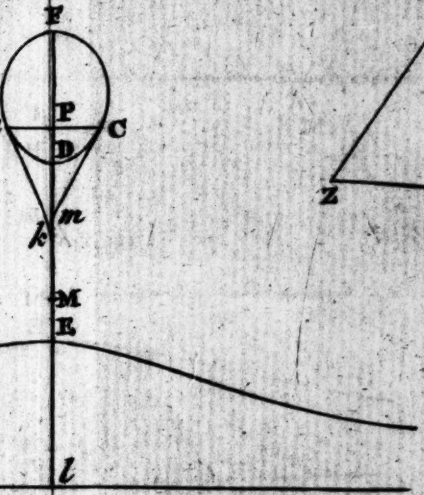
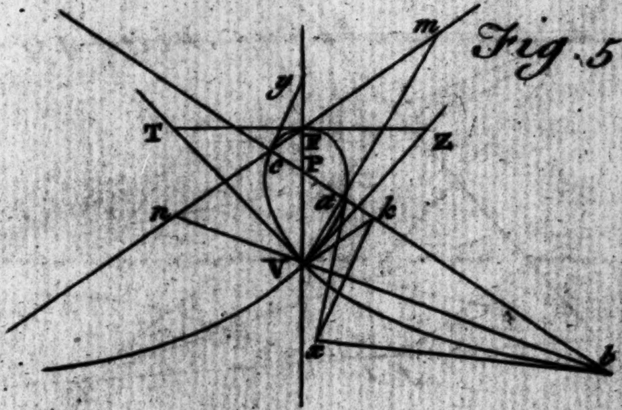


Fig. 50.



Appen. pa. 64.

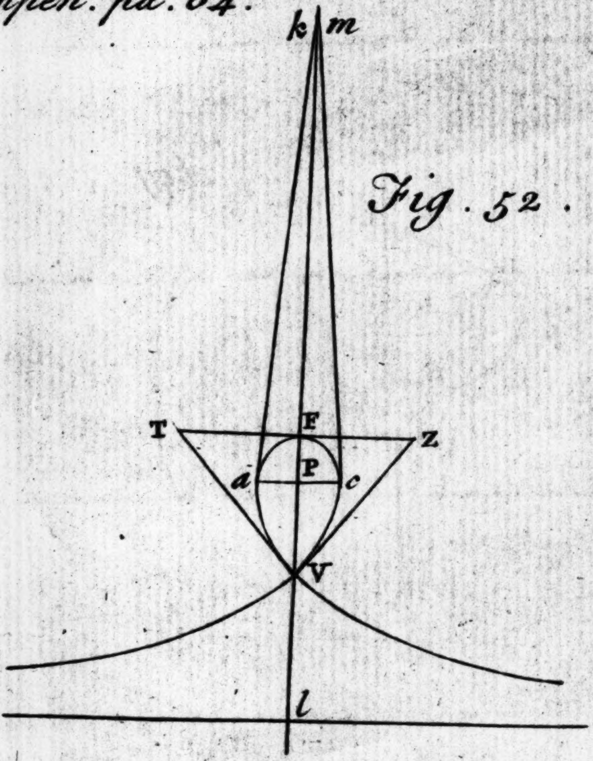


Fig. 52.

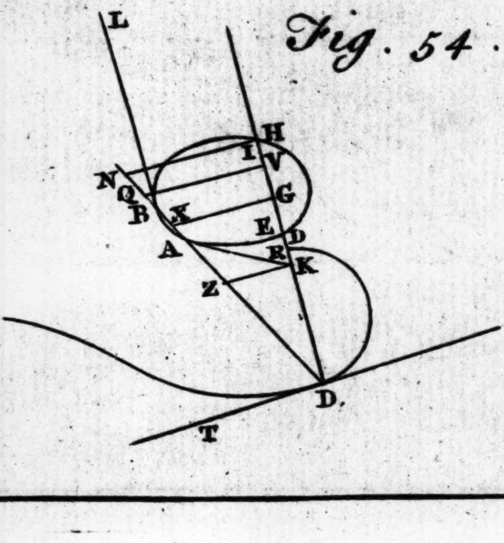


Fig. 54.

TAB. XI.

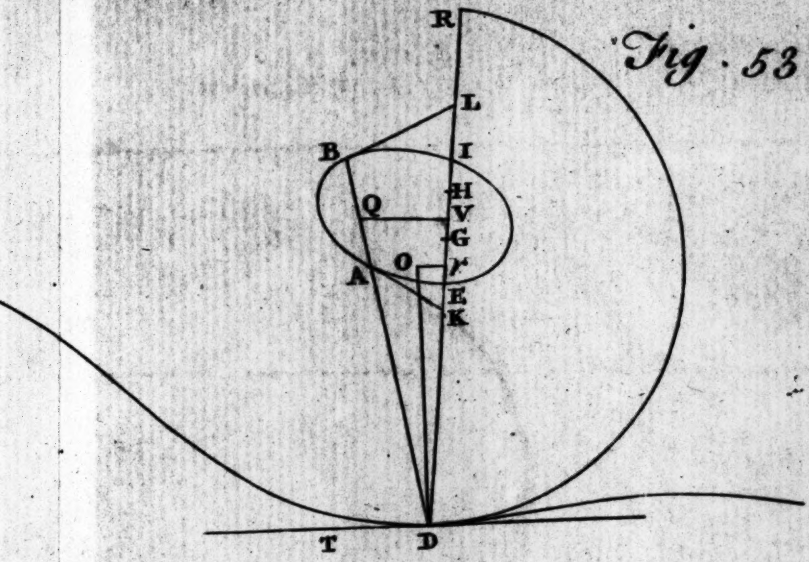


Fig. 53.

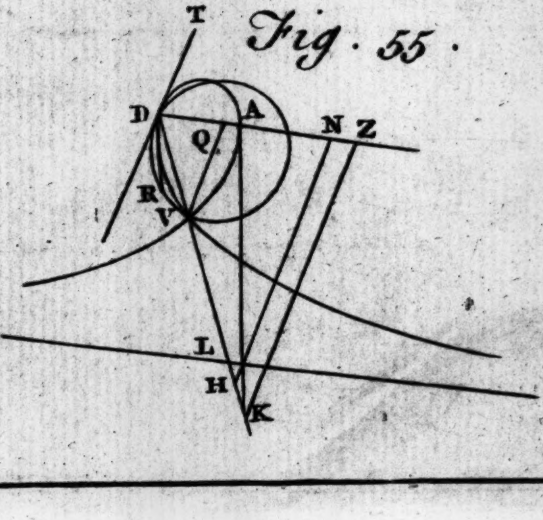


Fig. 55.

Appen. pa. 66.

Fig. 57.

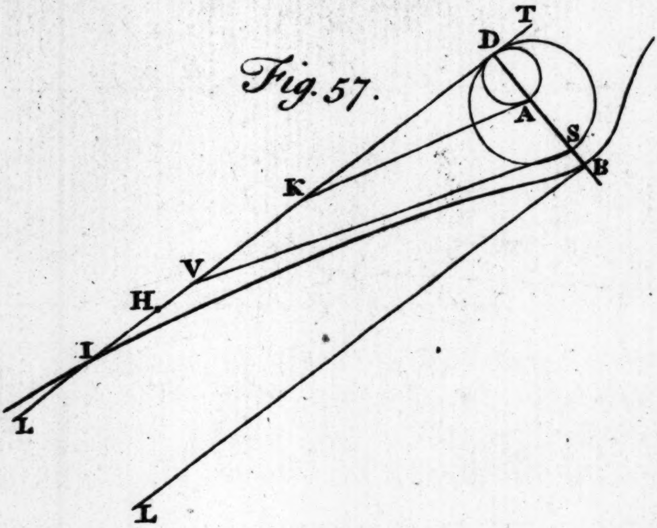
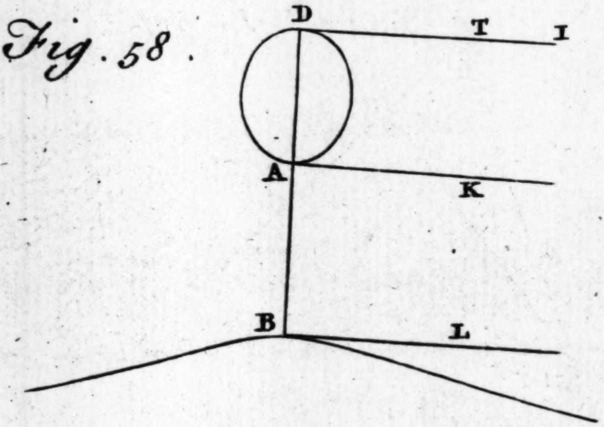
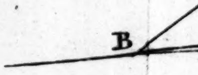


Fig. 58.





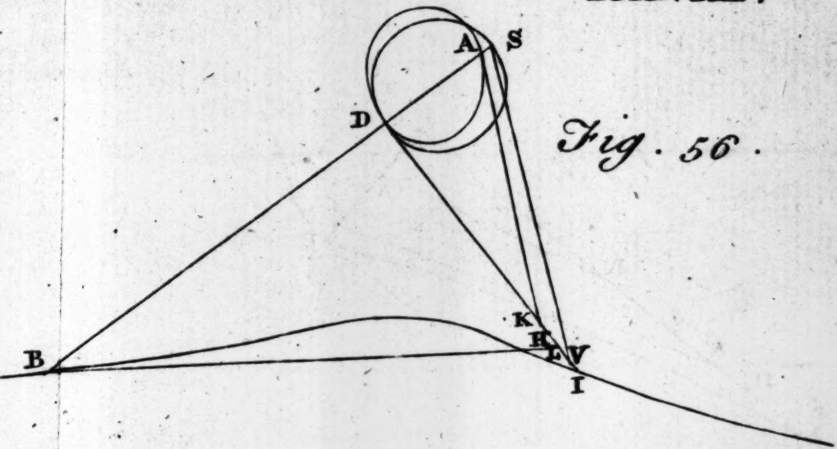


Fig. 56.

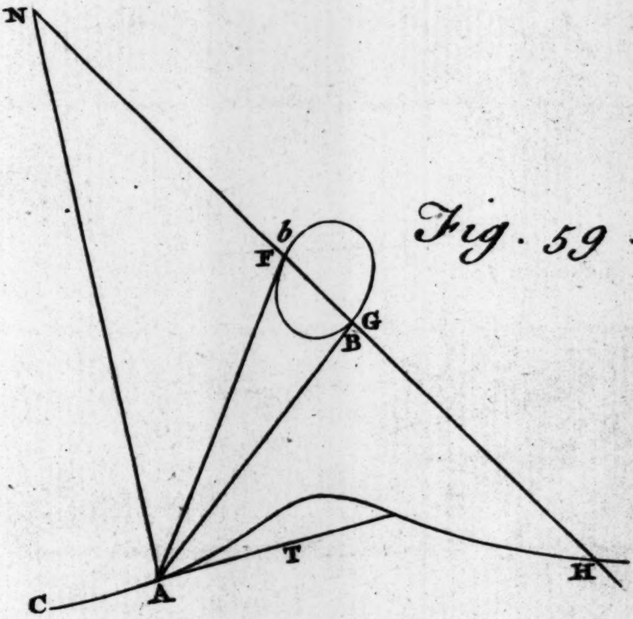


Fig. 59.

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